Minkowski’s second theorem over a Severi-Brauer variety

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Let $q$ be a positive definite quadratic form on $\mathbb{R}^n$ of discriminant $\text{disc}(q)$ and $\lambda_i(q)$, $i = 1, 2, \ldots, n$, the successive minima of $q$, i.e.,

$$\lambda_i(q) = \min\{\lambda > 0 : \{x \in \mathbb{Z}^n : q(x) \leq \lambda^2\} \text{ contains } i \text{ linearly independent vectors}\}.$$  

Then Minkowski’s second theorem asserts that the inequality

$$\lambda_1(q)\lambda_2(q)\cdots\lambda_n(q) \leq \gamma_n^{n/2}\text{disc}(q)^{1/2}$$  

holds, where $\gamma_n$ denotes Hermite’s constant.

Some generalizations of Minkowski’s second theorem were studied by Weyl, Mahler, Macfeat, Bombieri, Vaaler and Thunder, etc. Vaaler recently extended (1) to a twisted height on a vector space over an algebraic number field. To state Vaaler’s result, let $k$ denote an algebraic number field and $A$ the adele ring of $k$. For every $n$ by $n$ invertible matrix $g \in GL_n(A)$ with entries in $A$, the twisted height $H_g$ is defined on the $n$-dimensional vector space $k^n$. The successive minima $\lambda_i(g)$, $i = 1, 2, \ldots, n$, of $g$ are defined by

$$\lambda_i(g) = \min\{\lambda > 0 : \{x \in k^n : H_g(x) \leq \lambda\} \text{ contains } i \text{ linearly independent vectors}\}.$$  

Then, Vaaler proved the inequality

$$\lambda_1(g)\lambda_2(g)\cdots\lambda_n(g) \leq \gamma_n(k)^{n/2}|\det g|_A^{1/[k:Q]}.$$  

Here the constant $\gamma_n(k)$ is the generalized Hermite constant of $k$ defined by Icaza and Thunder. The inequality (2) coincides with (1) when $k = \mathbb{Q}$, $g = g_\infty$ (i.e., the finite adele part of $g$ is the identity) and the symmetric matrix corresponding to $q$ is equal to $g_\infty g_\infty$.

In my talk, I will show that Minkowski’s second theorem is extended to a free module over the matrix algebra $\mathfrak{A} = M_m(D)$, where $D$ is a central simple division algebra over a global field. We give a definition of the twisted heights on $\mathfrak{A}^n$ and introduce the generalized Hermite constant $\gamma_n(\mathfrak{A})$ of $\mathfrak{A}$. Then we obtain Minkowski’s second theorem for the successive minima of a given twisted height. Our theorem recovers (2) when $m = 1$ and $D = k$. Since the twisted height $H_g$ for $g \in GL_n(\mathfrak{A})$ is indeed a height on the projective space $\mathbb{P}^{n-1}(k)$, the inequality (2) is regarded as a statement on $\mathbb{P}^{n-1}(k)$. In this point of view, our result may be considered as enlargement of a base space from a projective space to a Severi-Brauer variety.