ON THE ZEROS OF THE SYMMETRIC SQUARE L-FUNCTION ASSOCIATED WITH THE RAMANUJAN DELTA-FUNCTION

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Abstract

Let \( \Delta(z) \) be the Ramanujan delta-function
\[
\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24} = \sum_{n=1}^{\infty} \tau(n) e^{2\pi inz}.
\]

For each \( p \) let \( \alpha_p \) and \( \beta_p \) be the roots of the polynomial \( X^2 - \tau(p)X + p^{11} \). Then the symmetric square \( L \)-function \( D_\Delta(s) \) attached to \( \Delta(z) \) is defined by the Euler product
\[
D_\Delta(s) = \prod_p ((1 - \alpha_p^2 p^{-s})(1 - \alpha_p \beta_p p^{-s})(1 - \beta_p^2 p^{-s}))^{-1}.
\]

The Euler product converges only when Re(\( s \)) is large, but there is a holomorphic continuation to the whole complex plane. It is well known that \( D_\Delta(z) \) has the integral expression
\[
\zeta^*(s)D_\Delta(s + 11) = \frac{(4\pi)^{s+11}}{\Gamma(s + 11)\Gamma(s + 11)} \int_{PSL_2(\mathbb{Z}) \setminus \mathbb{H}} y^{12 |\Delta(z)|^2} E^*(z, s) \frac{dx dy}{y^2},
\]
where \( \zeta^*(s) \) is the completed Riemann zeta function and \( E^*(z, s) \) is the completed non-holomorphic Eisenstein series for \( PSL_2(\mathbb{Z}) \). By using this integral expression and a property of the holomorphic projection of \( \Delta(z) \), we obtain the decomposition
\[
\tau(m)\zeta^*(s)D_\Delta(s + 11) = \frac{(4\pi)^{s+11}}{m^{11} \Gamma(11)\Gamma(s + 11)} \left\{ C_m(s) + R_m(s) \right\}
\]
for any \( m \geq 1 \), where
\[
C_m(s) = \tau(m) \left\{ \frac{\Gamma(s + 11)}{(4\pi m)^{s+11}} \zeta^*(2s) + \frac{\Gamma(12 - s)}{(4\pi m)^{12-s}} \zeta^*(2s - 1) \right\}
\]
and
\[
R_m(s) = \frac{\Gamma(s + 11)\Gamma(12 - s)}{2 (4\pi m)^{11}} \sum_{n=-m+1}^{\infty} \frac{\tau(m + n)}{(m + n)^{11/2}} |n|^{s-1} \sigma_{1-2s}(|n|) P_{s-1}^{-11} \left( \frac{2m + n}{n} \right).
\]

Now we denote by \( R_m^{\text{fin}}(s) \) a finite subseries of \( R_m(s) \).

In this talk we show that all zeros of \( C_m(s) + R_m^{\text{fin}}(s) \) lie on the line Re(\( s \)) = 1/2 except for finite ones.