On multiple zeta values and Bernoulli numbers

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Multiple zeta values \( \zeta(k) \) and \( \zeta^*(k) \), which are two of various natural generalizations of Riemann zeta values, are defined as follows. For any multiple index \( k = (k_1, k_2, \ldots, k_n) \) \( (k_i \in \mathbb{Z}, k_i > 0) \), the weight and height of \( k \) are by definition the integers \( k = k_1 + k_2 + \cdots + k_n \) and \( s = \#\{i \mid k_i > 1\} \), respectively. Index \( k = (k_1, k_2, \ldots, k_n) \) is said to be admissible if its first entry satisfies the extra requirement \( k_1 \geq 2 \). For each admissible multiple index \( k \), we define two kinds of multiple zeta values by

\[
\zeta(k) = \zeta(k_1, k_2, \ldots, k_n) = \sum_{m_1 > m_2 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}
\]

and

\[
\zeta^*(k) = \zeta^*(k_1, k_2, \ldots, k_n) = \sum_{m_1 \geq m_2 \geq \cdots \geq m_n \geq 1} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}.
\]

Multiple zeta values normally mean \( \zeta(k) \) in literatures, and Euler was interested in \( \zeta^*(k) \). In this talk, three families of relations between sums of multiple zeta values \( \zeta^* \) and Riemann zeta values are planning to be given. One of them is as follows.

**Theorem** (with Takashi Aoki) Let \( k \) and \( s \) be integers such that \( k/2 \geq s \geq 1 \). Let \( I_0(k, s) \) denote the set of all admissible multiple indices of weight \( k \) and height \( s \). Then we have

\[
\sum_{k \in I_0(k, s)} \zeta^*(k) = 2 \left( \frac{k - 1}{2s - 1} \right) (1 - 2^{1-k}) \zeta(k).
\]

Furthermore, a proof of certain kind of formula of Bernoulli numbers is given as a consequence of their theorems.