Asymptotic expansions of the non-holomorphic Eisenstein series II

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Abstract

Let $i = \sqrt{-1}$, $s = \sigma + it \in \mathbf{C}$ and H be the upper half plane. The non-holomorphic Eisenstein series for $SL_2(\mathbf{Z})$ with weight 0 is defined by

$$E(z,s) = y^s \sum_{\{c,d\}} |cz+d|^{-2s}.$$

Here $z = x + iy \in H$, and the summation is taken over $\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$, a complete system of representation of $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in SL_2(\mathbf{Z})\} \backslash SL_2(\mathbf{Z})$. The purpose of our talk is to describe an asymptotic formula of $E(z, \frac{1}{2} + it)$ when $t \to \infty$. In applications of the Rankin-Selberg method for the study of automorphic L-functions and the spectral theory, the growth condition of the Eisenstein series perform a fundamental role. It shoud be noted that the t-estimate of the Eisenstein series is a delicate problem. The non-constant terms of the Fourier expansion of $E(z,s)$ are not negligible. Our result may be considered as an analogy of Jutila's result on the square of the Riemann zeta-function, which is called Voronoi-Atkinson type formula.

In order to obtain our asymptotic formula, we need Olver's works based on the theory of asymptotic solutions of differential equations. The Airy function $\mathrm{Ai}(\tau)$ is crucial in the uniform asymptotic expantion of the Bessel function. Combining with the estimation of the sum of the divisor function, we have the following type formula.

$$\begin{aligned} E(z, \tfrac{1}{2} + it) \quad &= \tfrac{4\sqrt{2}\pi^{\frac{1}{2}+it} y^{\frac{1}{2}}}{\zeta(1+2it)} \sum_{n=1}^{N} n^{-it} \sigma_{2it}(n) \left\{ t^2 - (2\pi n y)^2 \right\}^{-\frac{1}{4}} \cos(2\pi n x) \tau_n^{\frac{1}{4}} \mathrm{Ai}(-\tau_n^{\frac{2}{3}}) \\ &+ \ (\text{constant term}) + (\text{error terms}). \end{aligned}$$

Here $\sigma_s(n)$ is the sum of s-th powers of positive divisors of n, and the parameter τ_n is defined by y, n and t. It is supposed that $t - (4\log t)^{\frac{2}{3}} t^{\frac{1}{3}} \le 2\pi y N \le t$.