An attempt to interpret the Weil explicit formula from Beurling’s spectral theory

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Let $\rho = \beta + i\gamma$ be the non-trivial zeros of the Riemann zeta-function. In a previous study we considered the quantity

$$\sum_{\rho} e^{u\rho^2-v\rho}, \quad u > 0, \quad v \in \mathbb{R},$$

and obtained asymptotic formulas as $u \to 0^+$. We saw that the quantity (1) behaves quite differently as $u \to 0^+$, according as $v = 0$, $v = \mp \log p^m$ ($p$ is a prime), or otherwise.

A motivation of considering the quantity (1) is as follows. To make the explanation simple, let us consider the quantity

$$\sum_{\rho} e^{u (\rho - 1/2)^2 - v (\rho - 1/2)}$$

instead of (1), and let us assume the Riemann hypothesis. Then this is equal to

$$\sum_{\gamma} e^{-u\gamma^2 - iv\gamma}.$$  

A. Beurling made great contributions to the problem of spectral synthesis. Roughly speaking, it is the problem for approximating a function $\phi$ by trigonometric polynomials in some topology. He introduced the transform

$$U_\phi(u, v) = \int_{-\infty}^{\infty} \phi(t)e^{-u|t|-ivt} dt, \quad u > 0, \quad v \in \mathbb{R}.$$  

According as the behavior for $U_\phi(u, v)$ as $u \to 0^+$, the concept of spectral sets is introduced. Roughly speaking, the spectral set of $\phi$ is defined to be the complement of the set of $v$ for which $U_\phi(u, v) \to 0$ as $u \to 0^+$. Spectral sets play a fundamental role in the problem of spectral synthesis.

We felt some resemblance between (2) and (3). This was a motivation of considering (1).

In this talk we will discuss a more general quantity than (1) and show asymptotic results for it.

This is a joint work with Masatoshi Suzuki.