On the functional equation of the Barnes multiple zeta-function

Masami Yoshimoto

Definition ([3, §15]). Let $r, i_k \in \mathbb{N}$, $\kappa = \sum_{k=1}^{r} i_k$, and let $\underline{\omega}$ satisfy following conditions:

$$\underline{\omega} = (\omega_1^{(1)}, \ldots, \omega_1^{(i_1)}, \ldots, \omega_r^{(1)}, \ldots, \omega_r^{(i_r)}),$$

where $\omega_k^{(m)}$ $(1 \le k \le r,\ 1 \le m \le i_k)$ satisfies

$$\begin{cases} |\omega_k^{(1)}| \le |\omega_k^{(2)}| \le \cdots \le |\omega_k^{(i_k)}|, \\ \arg \omega_k^{(1)} = \arg \omega_k^{(2)} = \cdots = \arg \omega_k^{(i_k)}, \\ \arg \omega_1^{(1)} < \arg \omega_2^{(1)} < \cdots < \arg \omega_r^{(1)}, \\ \arg \omega_r^{(1)} - \arg \omega_1^{(1)} < \pi, \end{cases}$$

and α is in the same side as $\underline{\omega}$ at the half-plane divided by straight line p which passes along the origin.

Then the Barnes multiple zeta-function is defined by

$$\zeta_\kappa(s, \alpha \mid \underline{\omega}) = \sum_{\underline{m} \in (\mathbb{N} \cup \{0\})^\kappa} \frac{1}{(\underline{m} \cdot \underline{\omega} + \alpha)^s} \quad (\operatorname{Re} s = \sigma > \kappa), \tag{1}$$

where $\underline{m} \cdot \underline{\omega}$ means an inner product of $\underline{m}$ and $\underline{\omega}$.

Theorem. *Let α satisfy either of conditions (a), (b), (c):*
(a) for all k $(1 \le k \le r)$,

$$\sum_{1 \le j < k} \sum_{m=1}^{i_j} \operatorname{Im} \frac{\omega_j^{(m)}}{\omega_k^{(1)}} < \operatorname{Im} \frac{\alpha}{\omega_k^{(1)}} < \sum_{k < j \le r} \sum_{m=1}^{i_j} \operatorname{Im} \frac{\omega_j^{(m)}}{\omega_k^{(1)}}.$$

(b) for some fixed k,

$$\alpha = c\omega_k^{(1)} + \sum_{1 \le j < k} \sum_{m=1}^{i_j} \omega_j^{(m)}, \quad 0 < c < \sum_{m=1}^{i_k} \frac{\omega_k^{(m)}}{\omega_k^{(1)}},$$

(c) *for some fixed* k,

$$\alpha = c\omega_k^{(1)} + \sum_{k<j\le r}\sum_{m=1}^{i_j}\omega_j^{(m)}, \quad 0 < c < \sum_{m=1}^{i_k}\frac{\omega_k^{(m)}}{\omega_k^{(1)}}.$$

And if $i_k \ge 2$ $(1 \le k \le r)$, *we suppose* $\omega_k^{(m)}/\omega_k^{(n)} \in \mathbb{R}\setminus\mathbb{Q}$ $(1 \le m < n \le i_k)$.

Then we have a functional equation

$$\begin{aligned} (2)\qquad & \zeta_\kappa(s,\alpha \mid \underline{\omega}) \\ &= \Gamma(1-s)2^{s+\kappa-1}\pi^{s-1}\sum_{k=1}^{r}\sum_{m=1}^{i_k}(\omega_k^{(m)})^{-s} \\ &\times \sum_{n=1}^{\infty}\frac{\cos\left(2\pi\frac{n}{\omega_k^{(m)}}\left(\alpha-\frac{1}{2}\sum_j\sum_l'\omega_j^{(l)}\right)+\frac{\pi}{2}(s-\kappa)\right)}{\prod_j\prod_l'\sin\left(\pi n\omega_j^{(l)}/\omega_k^{(m)}\right)}\cdot\frac{1}{n^{1-s}} \quad for\ \sigma<1, \end{aligned}$$

where $\sum_j\sum_l'$, $\prod_j\prod_l'$ *mean*

$$\sum_j\sum_l{}' := \sum_{\substack{j=1\\(j,l)\ne(k,m)}}^{r}\sum_{l=1}^{i_j}, \quad \prod_j\prod_l{}' := \prod_{\substack{j=1\\(j,l)\ne(k,m)}}^{r}\prod_{l=1}^{i_j}.$$

Especially if α *satisfies the condition* (a), *then the right-hand side of the functional equation* (2) *converges absolutely for* $s \in \mathbb{C}$. *If* α *satisfies conditions* (b) *or* (c) *for some fixed* k, *and if* $i_k = 1$ *or* $\omega_k^{(m)}/\omega_k^{(n)}$ *has a finite degree of irrationality in* $i_k \ge 2$ $(1 \le m < n \le i_k)$, *then the right-hand side of* (2) *converges absolutely for* $\sigma < 0$.

Corollary (**Arakawa [1, 2]**). *If* ξ *is an algebraic real irrational number, then a function* $\sum_{n=1}^{\infty}\frac{\cot\pi\xi n}{n^s}$ *converges absolutely for* $\mathrm{Re}\, s > 1$.

References

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