On the arithmetic distributions of simultaneous approximation convergents arising from Jacobi-Perron algorithm

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We fix a positive integer $d \geq 2$. Let $X = [0,1)^d$ with the Borel σ-algebra $\mathbb{B}$. Define a map $T : X \to X$ by

$$T\left((x_1, x_2, \ldots, x_d)\right) = \left(\frac{x_2}{x_1} - \left[\frac{x_2}{x_1}\right], \ldots, \frac{x_d}{x_1} - \left[\frac{x_d}{x_1}\right], \frac{1}{x_1} - \left[\frac{1}{x_1}\right]\right)$$

for $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in X$. Then there exists a unique absolutely continuous invariant probability measure μ. (X, T) is called the d-dimensional Jacobi-Perron algorithm. We put

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}^{(0)}(\mathbf{x}) = (k_1, k_2, \ldots, k_d) = \left(\left[\frac{x_2}{x_1}\right], \left[\frac{x_3}{x_1}\right], \ldots, \left[\frac{x_d}{x_1}\right], \left[\frac{1}{x_1}\right]\right) \quad \text{for } \mathbf{x} \in X$$

and

$$\mathbf{k}^{(s)}(\mathbf{x}) = \left(k_1^{(s)}, k_2^{(s)}, \ldots, k_d^{(s)}\right) = \mathbf{k}\left(T^{s-1}(\mathbf{x})\right) \quad \text{for} \quad s \geq 1.$$

We first define $Q^{(0)}$ as the $(d+1)\times(d+1)$ identity matrix I_{d+1}; then recursively $Q^{(n)}$ for $n \geq 1$ as

$$Q^{(n)} = Q^{(n-1)} \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & k_1^{(n)} \\ 0 & 1 & \ldots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & k_d^{(n)} \end{pmatrix}.$$

We set for $n \geq 1$

$$Q^{(n)} := \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \ldots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \ldots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \ldots & p_d^{(n-1)} & p_d^{(n)} \\ q^{(n-d)} & q^{(n-d+1)} & \ldots & q^{(n-1)} & q^{(n)} \end{pmatrix}.$$

*joint work with V. Berthé and H. Nakada

Then the sequence

$$\left\{ \left(\frac{p_1^{(k)}}{q^{(k)}}, \dots, \frac{p_d^{(k)}}{q^{(k)}} \right) : k \geq 1-d \right\}$$

is called the simultaneous approximation convergents of $\mathbf{x}$ from the d-dimensional Jacobi-Perron algorithm. It is well-known that for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$

$$\lim_{n\to\infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for} \quad 1 \leq i \leq d$$

holds.

Our result is that for almost every $\mathbf{x} \in X$ the sequences of vectors $\{(q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) : n \geq 1\}$ and $\{(p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) : n \geq 1\}$ are both equidistributed modulo m for any integer $m \geq 2$. More precisely: we put

$$\widetilde{\mathbb{Z}}_m^{d+1} = \{(\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1} : (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \text{ generates } \mathbb{Z}_m\}$$

and

$$c_m = \sharp\, \widetilde{\mathbb{Z}}_m^{d+1} \quad (\text{the cardinality of } \widetilde{\mathbb{Z}}_m^{d+1}).$$

One easily sees that

$$\begin{aligned} c_m &= \varphi_{d+1}(m) \\ &= \sharp\{(a_1, a_2 \dots, a_{d+1}) \in \{1, \dots, m\}^{d+1} : \gcd(a_1, a_2 \dots, a_{d+1}, m) = 1\}, \end{aligned}$$

where φ_{d+1} denotes the Jordan totient function of order $d+1$; we thus have

$$c_m = m^{d+1} \prod_{p|m} (1 - p^{-(d+1)}),$$

where the notation $\prod_{p|m}$ stands for the product over the prime numbers p that divide m. Then we have the following:

Main Theorem. *For almost every $\mathbf{x} \in X$, we have*

$$\begin{aligned} &\lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \ (\textit{mod. } m)\}}{N} \\ &= \lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N : (p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) \equiv (\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \ (\textit{mod. } m)\}}{N} \\ &= \frac{1}{c_m} = \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1} \prod_{p|m}(1 - p^{-(d+1)})} \end{aligned}$$

for any $(\alpha_1, \alpha_2 \dots, \alpha_{d+1}) \in \widetilde{\mathbb{Z}}_m^{d+1}$ with any integer $m \geq 2$.