Rational approximation of \((1 + x)^a\) and applications to the Thue inequality

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The Thue equation (inequality) is defined by

\[ F(X, Y) = k, \quad (\|F(X, Y)\| \leq k), \]

where \(F \in \mathbb{Z}[X, Y]\) is a homogeneous polynomial of degree \(n \geq 3\) and \(k\) is a positive integer. We have two fundamental theorems on them:

**Theorem T (A. Thue, 1909)** *Thue equation has only finitely many solutions \((X, Y) \in \mathbb{Z}^2\).*

This theorem is not effective. In 1968 A. Baker gave the upper bound of the solutions \((X, Y)\):

**Theorem B (A. Baker, 1968)** Let \(\kappa > n + 1\) and \((X, Y) \in \mathbb{Z}^2\) be a solution of the Thue equation. Then

\[
\max\{\|X\|, \|Y\|\} < C e^{(\log k)^{\kappa}},
\]

where \(C = C(n, \kappa, F)\) is an effectively computable number.

But generally this upper bound is too big to determine the exact solutions for a given equation (inequality).

Let \(\alpha\) and \(x\) be rational numbers in the open interval \((0, 1)\). We denote \(x = x_1/x_2\) and \(a = \alpha/n = a_1/a_2\) where \(x_1, x_2, a_1, a_2 \in \mathbb{Z}^+\) and \(\gcd(x_1, x_2) = \gcd(a_1, a_2) = 1\). By considering the integral

\[
I_n(x) = \frac{1}{2\pi\sqrt{-1}} \int \frac{z^n(1 + zx)^{n+a}}{(z(z-1))^{n+1}} \, dz,
\]

we have the Padé approximations to \((1 + x)^a\). By the Padé approximations we obtain

**Theorem 1** Let \(d = a_2^{1-(a_2 \mod 2)}, D = x_2, Q = a_2^2, p = (1+x)(1+x+\varepsilon x), P = \frac{1+x+\varepsilon x}{x(1-x)}, \varepsilon = \frac{1-x+\sqrt{1+3x+2x^2}}{2x}, L = \sin\frac{a\pi}{n}, l = \frac{1}{p^2}\) and

\[
\lambda = \frac{\log V}{\log U}, \quad C^{-1} = 2pdVC^\lambda,
\]

where \(V = PDQ, U = \frac{1}{2Q}\) and \(C = \max\{1, 2dl\}\). If \(x_2 > x_1^2a_2^{\frac{1}{2}}\) then

\[
\left| (1 + x)^a - \frac{X}{Y} \right| > \frac{c}{\sqrt{1+x}}
\]

for arbitrary positive integers \((X, Y)\).

Using Theorem 1 we obtain

**Theorem 2** Let \(k\) be a positive real number. Suppose \(x_2 > x_1^2a_2^{\frac{1}{2}}\), then any solution \((X, Y) \in (\mathbb{Z}^+)^2\) of the Thue inequality

\[
|X^n - (1 + x)^aY^n| \leq k
\]

satisfies

\[
Y < \begin{cases} 
\left(\frac{k}{nc}\right)^{\frac{1}{n}} & (X \geq Y), \\
\left(\frac{k}{(1+x)^a - 1}\right)^{\frac{1}{n}} & (X < Y).
\end{cases}
\]

This result depends on the method of G. V. Chudnovsky, J. H. Rickert and I. Wakabayashi. In the talk we shall show several examples.