Rational approximation of $(1+x)^a$ and applications to the Thue inequality

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The Thue equation (inequality) is defined by

$$F(X,Y)=k, \quad (|F(X,Y)|\le k),$$

where $F\in\mathbb{Z}[X,Y]$ is a homogeneous polynomial of degree $n\ge 3$ and k is a positive integer. We have two fundamental theorems on them:

Theorem T (A. Thue, 1909) *Thue equation has only finitely many solutions* $(X,Y)\in\mathbb{Z}^2$.

This theorem is not effective. In 1968 A. Baker gave the upper bound of the solutions (X,Y):

Theorem B (A. Baker, 1968) *Let* $\kappa>n+1$ *and* $(X,Y)\in\mathbb{Z}^2$ *be a solution of the Thue equation. Then*

$$\max\{|X|,|Y|\}<Ce^{(\log k)^\kappa},$$

where $C=C(n,\kappa,F)$ *is an effectively computable number.*

But generally this upper bound is too big to determine the exact solutions for a given equation (inequality).

Let α and x be rational numbers in the open interval $(0,1)$. We denote $x=x_1/x_2$ and $a=\alpha/n=a_1/a_2$ where $x_1,x_2,a_1,a_2\in\mathbb{Z}^+$ and $\gcd(x_1,x_2)=\gcd(a_1,a_2)=1$. By considering the integral

$$I_{in}(x)=\frac{1}{2\pi\sqrt{-1}}\int_\Gamma\frac{z^i(1+zx)^{n+a}}{(z(z-1))^{n+1}}\,dz,$$

we have the Padé approximations to $(1+x)^a$. By the Padé approximations we obtain

Theorem 1 *Let* $d={a_2}^{1-(a_2 \bmod 2)}$, $D=x_2$, $Q={a_2}^{\frac{3}{2}}$, $p=\frac{(1+\varepsilon)(1+x+\varepsilon x)}{1-\varepsilon}$, $P=\frac{1+x+\varepsilon x}{\varepsilon(1-\varepsilon)}$, $\varepsilon=\frac{-1-x+\sqrt{1+3x+2x^2}}{x}$, $l=\frac{\sin a\pi}{\pi}$, $L=\frac{1}{x^2}$ *and*

$$\lambda=\frac{\log V}{\log U}, \quad c^{-1}=2pdVC^\lambda,$$

where $V=PDQ$, $U=\frac{L}{DQ}$ *and* $C=\max\{1,2dl\}$. *If* $x_2>{x_1}^2{a_2}^{\frac{3}{2}}$ *then*

$$\left|(1+x)^a-\frac{X}{Y}\right|>\frac{c}{Y^{1+\lambda}}$$

for arbitrary positive integers (X,Y).

Using Theorem 1 we obtain

Theorem 2 *Let* k *be a positive real number. Suppose* $x_2>{x_1}^2{a_2}^{\frac{3}{2}}$, *then any solution* $(X,Y)\in(\mathbb{Z}^+)^2$ *of the Thue inequality*

$$|X^n-(1+x)^\alpha Y^n|\le k$$

satisfies

$$Y<\begin{cases}\left(\dfrac{k}{nc}\right)^{\frac{1}{n-1-\lambda}} & (X\ge Y),\\[2ex] \left(\dfrac{k}{(1+x)^\alpha-1}\right)^{\frac{1}{n}} & (X<Y).\end{cases}$$

This result depends on the method of G. V. Chudnovsky, J. H. Rickert and I. Wakabayashi. In the talk we shall show several examples.