Equivalent conditions for the algebraicity of the values of certain infinite products

Yohei Tachiya

In this talk, we will investigate the necessary and sufficient conditions for algebraicity of the values of certain infinite products.

Let K be an algebraic number field and $r \geq 2$ be integer. We define $\Omega_n \boldsymbol{z} = (z_1^{r^n}, \ldots, z_m^{r^n})$ for $\boldsymbol{z} = (z_1, \ldots, z_m)$ and put

$$\Phi_0(\boldsymbol{z}) = \prod_{k=0}^{\infty} \frac{E_k(\Omega_k \boldsymbol{z})}{F_k(\Omega_k \boldsymbol{z})},$$

where $E_k(\boldsymbol{z})$ and $F_k(\boldsymbol{z})$ are polynomials in $K[\boldsymbol{z}]$ such that the degrees are bounded and the coefficients satisfy suitable conditions. Suppose that there exists a positive integer D such that $DF_k(\boldsymbol{z})$ $(k \geq 0)$ are the polynomials with integer coefficients of K. Then the main theorem can be stated as in the following;

Main theorem. *Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in K^m$ be an algebraic point with $0 < |\alpha_i| < 1$ $(1 \leq i \leq m)$ such that $|\alpha_1|, \ldots, |\alpha_m|$ are multiplicatively independent and $E_k(\Omega_k \boldsymbol{\alpha}) F_k(\Omega_k \boldsymbol{\alpha}) \neq 0$ $(k \geq 0)$. Then $\Phi_0(\boldsymbol{\alpha})$ is algebraic if and only if $\Phi_0(\boldsymbol{z})$ is a rational function with coefficients in K.*

As applications of the main theorem, we obtain the following results.
i) Let $\{a_n^{(i)}\}_{n \geq 0}$ $(1 \leq i \leq m)$ be m sequences in K satisfying suitable conditions and

$$\Phi_0(\boldsymbol{z}) = \prod_{k=0}^{\infty} \left(1 + a_k^{(1)} z_1^{r^k} + \cdots + a_k^{(m)} z_m^{r^k}\right),$$

where $a_n^{(1)} \neq 0$ for infinitely many n. Let $\boldsymbol{\alpha}$ be an algebraic point as in the main theorem. Then $\Phi_0(\boldsymbol{\alpha})$ is algebraic if and only if $r = 2$, $a_n^{(i)} = 0$ $(i \neq 1)$, and there exists a root of unity ω such that $a_n^{(1)} = \omega^{2^n}$ for every large n.
ii) The number

$$\prod_{k=0}^{\infty} \left(1 + \frac{a_k}{F_{r^k}}\right)$$

is transcendental, where F_n is n-th Fibonacci number and $\{a_n\}_{n \geq 0}$ is suitable sequence of algenraic numbers.