Equivalent conditions for the algebraicity of the values of certain infinite products

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In this talk, we will investigate the necessary and sufficient conditions for algebraicity of the values of certain infinite products.

Let $K$ be an algebraic number field and $r \geq 2$ be integer. We define $\Omega_n z = (z_1^n, \ldots, z_m^n)$ for $z = (z_1, \ldots, z_m)$ and put

$$\Phi_0(z) = \prod_{k=0}^{\infty} \frac{E_k(\Omega_k z)}{F_k(\Omega_k z)},$$

where $E_k(z)$ and $F_k(z)$ are polynomials in $K[z]$ such that the degrees are bounded and the coefficients satisfy suitable conditions. Suppose that there exists a positive integer $D$ such that $DF_k(z) (k \geq 0)$ are the polynomials with integer coefficients of $K$. Then the main theorem can be stated as in the following:

**Main theorem.** Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in K^m$ be an algebraic point with $0 < |\alpha_i| < 1 (1 \leq i \leq m)$ such that $|\alpha_1|, \ldots, |\alpha_m|$ are multiplicatively independent and $E_k(\Omega_k \alpha)F_k(\Omega_k \alpha) \neq 0 (k \geq 0)$. Then $\Phi_0(\alpha)$ is algebraic if and only if $\Phi_0(z)$ is a rational function with coefficients in $K$.

As applications of the main theorem, we obtain the following results.

i) Let $\{a^{(i)}_n\}_{n \geq 0} (1 \leq i \leq m)$ be $m$ sequences in $K$ satisfying suitable conditions and

$$\Phi_0(z) = \prod_{k=0}^{\infty} \left( 1 + a^{(1)}_k z_1^k + \cdots + a^{(m)}_k z^k_m \right),$$

where $a^{(1)}_n \neq 0$ for infinitely many $n$. Let $\alpha$ be an algebraic point as in the main theorem. Then $\Phi_0(\alpha)$ is algebraic if and only if $r = 2$, $a^{(i)}_n = 0 (i \neq 1)$, and there exists a root of unity $\omega$ such that $a^{(1)}_n = \omega^{2n}$ for every large $n$.

ii) The number

$$\prod_{k=0}^{\infty} \left( 1 + \frac{a_k}{F_{rk}} \right)$$

is transcendental, where $F_n$ is $n$-th Fibonacci number and $\{a_n\}_{n \geq 0}$ is suitable sequence of algebraic numbers.