Algebraic independence of a certain series and its subseries with subscripts in a geometric progression

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The main result of this talk asserts the algebraic independence of $\sum_{n=1}^{\infty} a_n$ and its subseries $\sum_{n=1}^{\infty} a_{d^n}$, where $\{a_n\}_{n\geq 1}$ is a sequence of rational numbers such that $\sum_{n=1}^{\infty} a_n$ absolutely converges and d is an integer greater than 1.

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$. Rabinowitz [2] proved that for every $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k}} = \frac{1}{F_{2k}} \sum_{n=1}^{k} \frac{1}{F_{2n-1} F_{2n}}.$$

In this talk we consider the similarly constructed series such as $\displaystyle\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}}$ $(k \in \mathbb{N})$, where $[x]$ denotes the largest integer not exceeding the real number x. These sums are not only transcendental but also algebraically independent. For example, the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{d^n} F_{d^n+2k}} \quad (k \in \mathbb{N})$$

are algebraically independent. This result is proved by using Mahler's method with linear relations between the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{d^n} F_{d^n+k}} \quad (k \in \mathbb{N}).$$

It seems difficult to find in literature the results which assert the algebraic independence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{d^n}$ mentioned above. For example, the algebraic independency of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \geq 3)$ is open, while Nishioka, Tanaka, and Toshimitsu [1] proved that the numbers $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \geq 3)$ are algebraically independent.

References

[1] K. Nishioka, T. Tanaka, and T. Toshimitsu, *Algebraic independence of sums of reciprocals of the Fibonacci numbers*, Math. Nachr. **202** (1999), 97–108.

[2] S. Rabinowitz, *Algorithmic summation of reciprocals of products of Fibonacci numbers*, Fibonacci Quart. **37** (1999), 122–127.