NEW SERIES TRANSFORMATIONS FOR EULER’S CONSTANT

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Abstract

Let $s_n = 1 + 1/2 + \ldots + 1/(n-1) - \log n$. In 1995, the author has found a series transformation of the type $\sum_{k=0}^n \mu_{n,k,\tau} s_{k+\tau}$ with integer coefficients $\mu_{n,k,\tau}$, from which geometric convergence to Euler’s constant $\gamma$ for $\tau = O(n)$ results. In recently published papers T. Rivoal and Kh. and T. Hessami Pilehrood have generalized this result. In the lecture a series transformation $\sum_{k=0}^n \mu_{n,k,\tau_1,\tau_2} s_{k+\tau_1} + \tau_2$ with two parameters $\tau_1$ and $\tau_2$ satisfying $\tau_1 + 1 \leq \tau_2 \leq n + \tau_1 + 1$ and integer coefficients $\mu_{n,k,\tau_1}$ will be introduced. By applying the Mellin-Barnes integral representation of the $3F_2$-function, combinatorial identities and the analysis of the $\psi$-function, for $n = 2m$, $\tau_1 = m - 1$ and $\tau_2 = 2m$ it is shown that $S := |\sum_{k=0}^n \mu_{n,k,\tau_1} s_{k+\tau_2} - \gamma| \leq m/2 \cdot |\zeta(2) - q_m|$, where $q_m$ are explicitly given rational numbers. Finally, $\zeta(2) - q_m$ can be expressed in terms of Legendre-type integrals, which give upper bounds for $S$. In particular, for $n = 2m$, $\tau_1 = m - 1$ and $\tau_2 = 2m$ this bound equals to $2m \cdot 64^{-m}$. By the way, we find a linear three-term recurrence formula for a specific weighted sum on a $3F_2$-function.