Diophantine Analysis and Related Fields 2019
Abstracts of the talks

Thursday 7th March

10:00–10:50 **Makoto Kawashima** (Osaka University)
TITLE: **Linear independence measures for values of power of logarithm functions**
Abstract: In this talk, we construct (type I) Padé approximations of power of logarithm functions using that of exponential functions. Applying these approximations, we give new linear independence measures of values of logarithm functions at algebraic numbers with sufficiently small absolute value in both complex and p-adic cases. If time permits, we also discuss q-analogue of these results.

11:00-11:50 **Alan Filipin** (University of Zagreb)
TITLE: **A polynomial variant of the problem of Diophantus**
Abstract: In this talk we prove that if $\{a, b, c, d\}$ is a set of four non-zero elements of $\mathbb{R}[X]$, not all constant, such that the product of any two of its distinct elements increased by 1 is a square of an element of $\mathbb{R}[X]$, then

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1).$$

Some consequences of the above result are that for an arbitrary positive integer n there does not exist a set of five non-zero elements from $\mathbb{Z}[X]$, which are not all constant, such that the product of any two of its distinct elements increased by n is a square of an element of $\mathbb{Z}[X]$. Furthermore, there can exist such a set of four non-zero elements of $\mathbb{Z}[X]$ if and only if n is a square.

Moreover, we prove the same result for the polynomials with coefficients from the ring of Gaussian integers.

This is joint work with Ana Jurasić.

14:00-14:50 **Yasutsugu Fujita** (Nihon University) (joint work with Florian Luca)
TITLE: **Non-existence of Diophantine quadruples consisting of Fibonacci numbers**
Abstract: A set $\{a_1, \ldots, a_m\}$ of m positive integers is called *a Diophantine m-tuple* if $a_i a_j + 1$ is a perfect square for all i, j with $i < j$. The simplest Diophantine quadruple $\{1, 3, 8, 120\}$ was found by Fermat. Baker and Davenport proved in 1969 that the extension of the triple $\{1, 3, 8\}$ to a Diophantine quadruple must be $\{1, 3, 8, 120\}$. Various kinds of generalizations of this result exist. One of them states that if $\{F_{2n}, F_{2n+2}, F_{2n+4}, d\}$ is a Diophantine

quadruple with n a positive integer and F_k the kth Fibonacci number, then $d = 4F_{2n+1}F_{2n+2}F_{2n+3}$, which was shown by Dujella in 1999. In this direction, He, Luca and Togbé showed in 2016 that if $\{F_{2n}, F_{2n+2}, F_k\}$ is a Diophantine triple with $n > 1$, then $k = 2n - 2$ or $2n + 4$. They also conjectured that there are no Diophantine quadruples consisting of Fibonacci numbers. In 2017, we showed that there are only finitely many Diophantine quadruples consisting of Fibonacci numbers. In this talk, we give a proof of the conjecture above. The key to the proof is to apply Baker's method on linear forms in logarithms repeatedly to an equation satisfied by a binary recurrence sequence and the Fibonacci sequence.

15:00-15:50 **Thomas Stoll** (Université de Lorraine)
TITLE: **The sum of digits in two different bases**
Abstract: Let $s_a(n)$ denote the sum of digits of an integer n in the base a expansion. We show that, provided a and b are multiplicatively independent integers, any positive real number is a limit point of the sequence $\{s_b(n)/s_a(n)\}_n$. We also provide upper and lower bounds for the counting functions of the corresponding subsequences. The proofs involve exponential sums, discrepancy estimates as well as the irrationality measure of $\log a/\log b$. This is joint work with R. de la Bretèche and G. Tenenbaum.

16:00-16:50 **Ryuji Abe** (Tokyo Polytechnic University)
TITLE: **Markoff spectrum and arithmetic Fricke groups**
Abstract: The Markoff spectrum is defined as the set of the normalized values of arithmetic minima of indefinite quadratic forms. Fricke groups are two-generator free Fuchsian groups the quotient spaces of which are once punctured tori. It is well-known that the discrete part of the Markoff spectrum is interpreted in terms of simple closed geodesics on the once punctured torus of an arithmetic Fricke group which is a subgroup of the modular group with index 6. In this talk, we report an interpretation of sequences in not discrete part of the Markoff spectrum using other arithmetic Fricke groups. We use a graph representation of Diophantine equations whose integer solutions give the sequences in the Markoff spectrum. This talk is based on joint work with I. R. Aitchison and B. Rittaud.

17:00-17:50 **Iekata Shiokawa** (Keio University) (joint work with Daniel Duverney and Takeshi Kurosawa)
TITLE: **Irrationality exponents of certain sums of rational numbers**
Abstract: Let $\{x_n\}_{n\geq 0}$ be a sequence of rational numbers greater than one such that $z_{n+1} = x_{n+1}x_n^{-2} \geq 1$ for all sufficiently large n and let $\varepsilon_n \in \{-1, 1\}$ with $\varepsilon_1 = 1$. Define $\delta_1 = \operatorname{den} x_1$ and $\delta_{n+1} = \delta_n^2 \operatorname{den} z_{n+1}$ $(n \geq 1)$, where $\operatorname{den}(a/b) = |b|$ if $a, b \in \mathbb{Z}$. Assume that $\log \delta_{n+1} = o(\log x_n)$. Then the irrationality exponent of the number $\sum_{n=1}^{\infty} \varepsilon_n/x_n$ is equal to $\limsup_{n\to\infty} \log x_{n+1}/\log x_n$, where the irrationality exponent of irrational number α is defined by the supremum of the set of numbers μ for which the inequality $|\alpha - p/q| < q^{-\mu}$ has infinitely many rational solutions p/q with $q > 0$.

Friday 8th March

9:00-9:50 **Yohei Tachiya** (Hirosaki University)
TITLE: **Refinement of Chowla-Erdős method and arithmetical properties of certain Lambert series**
Abstract: In this talk, we refine the method of Chowla and Erdős on the irrationality of Lambert series. As applications of our main theorem, we give linear independence results for various kinds of Lambert series. Moreover, we show the irrationality of certain reciprocal sums of Fibonacci and Lucas numbers associated with prime numbers. This is a joint work with Daniel Duverney and Yuta Suzuki.

10:00-10:50 **Dong Han Kim** (Dongguk University)
TITLE: **On the set of inhomogeneous Diophantine approximations**
Abstract: Fix an irrational number θ. We consider the set of points y such that $\|n\theta - y\| < \varphi(n)$ with monotone error functions $\varphi(n)$. We give a necessary and sufficient condition for the set has full Lebesgue measure and calculate the Hausdorff dimension. We also consider the dimension of the uniform inhomogeneous approximation and the dimension of badly approximable sets, which depend on the Diophantine property of the irrational number θ. This talk is based on joint works with Yann Bugeaud, Michael Fuchs, Lingmin Liao, Seonhee Lim, Michal Rams and Baowei Wang.

11:00-11:50 **Teturo Kamae** (Osaka City University)
TITLE: **Normal numbers as configurations on $\mathbb{N}^2$**
Abstract: Let $\mathbb{A} = \{0, 1, \cdots, d-1\}$ be an alphabet. We consider a configuration $x \in \mathbb{A}^{\mathbb{N}^2}$, where $\mathbb{N} = \{1, 2, \cdots\}$. We denote its restriction to a rectangle $C \times D \subset \mathbb{N}^2$ by $x[C \times D] \in \mathbb{A}^{C\times D}$. We denote its translation by $(i, j) \in (\mathbb{N} \cup \{0\})^2$ as $T^{(i,j)}x[C \times D] \in \mathbb{A}^{(C+i)\times(D+j)}$. That is,

$$T^{(i,j)}x[C \times D](u + i, v + j) = x[C \times D](u, v)$$

for any $(u, v) \in C \times D$. Let

$$\mathbb{A}^+ = \bigcup_{n=1,2,\cdots;m=1,2,\cdots} \mathbb{A}^{[n]\times[m]},$$

where we denote $[n] = \{1, 2, \cdots, n\}$. For $\eta \in \mathbb{A}^{C\times D}$ and $\xi \in \mathbb{A}^{[n]\times[m]}$, we say that ξ is a (i, j)-prefix of η denoting $\xi \prec_{(i,j)} \eta$ if $([n]+i)\times([m]+j) \subset C\times D$ and $\eta[([n] + i) \times ([m] + j)] = T^{(i,j)}\xi$. We denote by $|\eta|_\xi$ the number of occurrences of ξ in η. That is,

$$|\eta|_\xi = \#\{(i, j);\ \xi \prec_{(i,j)} \eta\}.$$

We call $x \in \mathbb{A}^{\mathbb{N}^2}$ *normal* if

$$\lim_{N,M\to\infty} \frac{1}{NM}|x[[N] \times [M]]|_\xi = d^{-nm}$$

holds for any $n, m \in \mathbb{N}$ and $\xi \in \mathbb{A}^{[n]\times[m]}$. Let X be a $\mathbb{A}^{\mathbb{N}^2}$-valued random variable such that $\{X(n, m);\ (n, m) \in \mathbb{N}^2\}$ are i.i.d. random variables with $P(X(0, 0) = a) = 1/d$ for any $a \in \mathbb{A}$. Then, almost all realizations $x \in \mathbb{A}^{\mathbb{N}^2}$ of X are normal.

We consider a stronger notion than normality. For $x \in \mathbb{A}^{\mathbb{N}^2}$, we define

$$\Sigma^{NM}(x) = \sum_{\xi \in \mathbb{A}^+} |x[[N] \times [M]]|_\xi^2.$$

Then, almost all realizations $x \in \mathbb{A}^{\mathbb{N}^2}$ of X satisfy that

$$\lim_{N,M\to\infty} \frac{1}{N^2 M^2} \Sigma^{NM}(x) = \frac{1}{4} + \sum_{k=1}^{\infty} \frac{1}{d^k - 1}.$$

We call $x \in \mathbb{A}^{\mathbb{N}^2}$ as this Σ-random. Then, Σ-random implies normal, but the converse is not true. We study the normality and the Σ-randomness in this setting.

14:00-14:50 **Alan Haynes** (University of Houston)
TITLE: **Diophantine approximation and diffraction from quasicrystals**
Abstract: In this talk we will explain how diffraction patterns observed from cut and project sets (models for physical materials called quasicrystals) are determined by Diophantine approximation properties of the underlying constructions. The classical approach to calculating diffraction patterns seen from these objects assumes an infinite model, and for this reason it is not the most practical, from an experimental point of view. Our approach (joint work with Michael Baake) quantifies precisely how much the diffraction patterns observed from finite patterns in cut and project sets deviate from the infinite models. Our methods are explicit and geared towards numerical computation, and they demonstrate the importance of Diophantine approximation to accurately determining complex phases and amplitudes of these diffraction images.

15:00-15:50 **Hohto Bekki** (The University of Tokyo)
TITLE: **On the geodesic continued fraction and its periodicity**
Abstract: The classical Lagrange theorem in the theory of continued fraction says that (1) the continued fraction expansion of a given real number becomes periodic if and only if the number is a real quadratic irrational, and (2) the period of the continued fraction expansion actually gives the fundamental unit of the associated (order of) real quadratic field. On the other hand, we know that (1) a geodesic on the upper-half plane becomes a closed geodesic ("periodic") on the modular curve if and only if the two endpoints are the conjugate real quadratic irrationals, and (2) the length of the closed geodesic becomes the regulator of the associated (order of) real quadratic field. Based on this analogy, we have studied generalizations of continued fraction using the geodesics on some locally symmetric spaces, and established the generalizations of the Lagrange theorem. In this talk we mainly discuss the case where the locally symmetric space is the Shimura curve coming from the (2,3,7)-triangle group.

16:00-16:50 **Arnaldo Nogueira** (Institut de Mathématiques de Marseille)
TITLE: **On the action of the semigroup of non singular integral matrices on $\mathbb{R}^n$**
Abstract: Let Γ be the multiplicative semigroup of all $n \times n$ matrices with integral entries and nonzero determinant. Let $1 \leq p \leq n-1$ and $V = \mathbb{R}^{np} =$

$\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n$ (p copies). Consider the action of Γ on V, given by the natural action on each component, by matrix multiplication on the left. Then for $\mathbf{x} = (x_1, \ldots, x_p) \in V$, the Γ-orbit is dense in V if and only if there is no linear combination $\sum_{j=1}^{p} \lambda_j x_j$, with $\lambda_j \neq 0$ for some j, which is a rational vector in $\mathbb{R}^n$; in fact the assertion holds also for the orbit of the group $\mathrm{SL}(n, \mathbb{Z})$ that is contained in Γ. When the Γ-orbit of $\mathbf{x}$ is dense, given $\mathbf{y} \in \mathbb{R}^{np}$, and $\epsilon > 0$ one may ask for $\gamma \in \Gamma$ such that $\|\gamma\mathbf{x} - \mathbf{y}\| < \epsilon$, with a bound on $\|\gamma\|$ in terms of ϵ. There has been considerable interest in the literature in quantitative results of this kind, for various group actions. In particular it was shown by Laurent-Nogueira, for $n = 2$, that given an irrational vector $\mathbf{x}$ in $\mathbb{R}^2$, any target vector $\mathbf{y} \in \mathbb{R}^2$ and $\rho < \frac{1}{3}$ there exist infinitely many γ in $\mathrm{SL}(2, \mathbb{Z})$ such that $\|\gamma\mathbf{x} - \mathbf{y}\| \leq \|\gamma\|^{-\rho}$. In the talk we will describe some results along this theme for the action of Γ; for the case $n = 2$ the result is stronger in import than what is recalled above for $\mathrm{SL}(2, \mathbb{Z})$, in the sense that the corresponding statement holds for all ρ less than 1, in place of $\frac{1}{3}$ for $\mathrm{SL}(2, \mathbb{Z})$.

The talk is based on a joint work with S.G. Dani.

17:00-17:50 **Haruki Ide** (Keio University)
TITLE: **Algebraic independence of the values and the partial derivatives of a certain entire function of two variables**
Abstract: In previous works, Tanaka constructed a certain entire function with the property that its values and its derivatives of any order at any distinct nonzero algebraic numbers are algebraically independent. He also constructed an entire function defined by an infinite product and having the property that its values and its derivatives of any order at any distinct nonzero algebraic numbers except zeroes of the infinite product are algebraically independent. In this talk, the speaker will introduce a certain entire function of two variables with the following remarkable properties:
(1) All the numbers whose algebraic independency was proved in Tanaka ' s previous works stated above are given as the values or the partial derivatives of this two-variable function at algebraic points.
(2) The infinite set consisting of the values and the partial derivatives of any order of this two-variable function at any distinct algebraic points is algebraically independent.

Saturday 9th March

10:00-10:50 **Tomohiro Ooto** (Tecnos Data Science Engineering)
TITLE: **Diophantine exponents for hyperquadratic continued fractions**
Abstract: Hyperquadratic continued fractions are some class of algebraic continued fractions in the field of Laurent series over a finite field. Many explicit continued fractions are known for nonquadratic but hyperquadratic elements. In this talk, we estimate Diophantine exponents w_n and w_n^* for such hyperquadratic continued fractions. As its applications, we determine the degrees of such hyperquadratic continued fractions. This is a joint work with K. Ayadi (Univ. of Sfax).

11:00-11:50 **Takafumi Miyazaki** (Gunma University)
TITLE: **Application of cubic residue theory to a special type of unit equation concerning Eisenstein triples**
Abstract: A special type of unit equation is of the form $a + b = c$, where a, b, c are pair-wise coprime positive integer variables satisfying the condition that any prime dividing abc belongs to a given finite set S of primes. As an application to the celebrated theorem of Thue-Siegel-Roth, it was shown that the mentioned unit equation has only finitely many solutions (a, b, c), and the number of such triples is at most an explicit constant depending only on S. Though the theory of linear forms in complex logarithms of Baker can be applied to estimate the size of possible triples (a, b, c), it is still not easy to describe all such triples even for very special cases.

In this talk, as a very special case of the mentioned unit equation, we treat the Diophantine equation $A^x + B^y = C^z$, where A, B, C are given integers with $\min\{A, B, C\} > 1$ and $\gcd(A, B, C) = 1$. We solve this equation for some family of triples (A, B, C) such that A, B, C forms three length of a triangle having a $2\pi/3$ angle. A key of the proof is to compute values of cubic residue characters. The result can be regarded as a relevant analogue to some results on Jeśmanowicz' conjecture concerning primitive Pythagorean triples.

12:00-12:50 **Hajime Kaneko** (University of Tsukuba)
TITLE: **Analogy of the Lagrange spectrum for powers of quadratic Pisot units**
Abstract: For a real number x, let $\|x\|$ be the distance of x and its nearest integer. The Lagrange spectrum is related to the Diophantine approximation property of the arithmetic sequences $(n\alpha)_{n=0,1,\ldots}$. More precisely, for any irrational number α, let $L(\alpha) = 1/c_\alpha$, where $c_\alpha = \liminf_{n\to\infty} n\|n\alpha\|$. Then the Lagrange spectrum is defined by $\mathcal{L} = \{L(\alpha) \mid \alpha \in \mathbb{R}\backslash\mathbb{Q}\}$.

In this talk we consider analogy of the Lagrange spectrum for geometric sequences. Little is known about Diophantine approximation properties of geometric sequences. For instance, it is still open problem to disprove that $\lim_{n\to\infty} \|e^n\| = 0$. The main purpose of this talk is to investigate the set $\{\limsup_{n\to\infty} \|\xi\alpha^n\| \mid \xi \in \mathbb{R}\}$, where α is a quadratic Pisot unit.

This is a joint work with Shigeki Akiyama and Teturo Kamae.