SCHEDULED TALKS AND THEIR ABSTRACTS FOR "DIOPHANTINE ANALYSIS AND RELATED FIELDS 2011"

Martin Stein (FHDW Hannover)

Title: *Algebraic Independence Results for Reciprocal Sums of Fibonacci and Lucas Numbers*

Abstract: Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively. D. Duverney, Ke. Nishioka, Ku. Nishioka and I. Shiokawa proved that the values of the Fibonacci zeta function $\zeta_F(2s) = \sum_{n=1}^{\infty} F_n^{-2s}$ are transcendental for any $s \in \mathbb{N}$ using Nesterenko's theorem on Ramanujan functions $P(q)$, $Q(q)$, and $R(q)$. They obtained similar results for the Lucas zeta function $\zeta_L(2s) = \sum_{n=1}^{\infty} L_n^{-2s}$ and some related series. Later, C. Elsner, S. Shimomura and I. Shiokawa found conditions for the algebraic independence of these series. In my PhD thesis I generalized their approach and treated the following problem: We investigate all subsets of

$$\left\{ \sum_{n=1}^{\infty} \frac{1}{F_n^{2s_1}}, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n^{2s_2}}, \sum_{n=1}^{\infty} \frac{1}{L_n^{2s_3}}, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{L_n^{2s_4}} \;:\; s_1, s_2, s_3, s_4 \in \mathbb{N} \right\},$$

and decide on their algebraic independence over $\mathbb{Q}$. Actually this is a special case of a more general theorem for reciprocal sums of binary recurrent sequences.

Carsten Elsner (FHDW Hannover)

Title: *New Algebraic Independence Results Based on a Theorem of Y. Nesterenko*

Abstract: In their joint work during the last years the authors established a method how to decide on the algebraic independence of a set $\{y_1, \ldots, y_n\}$ when these numbers are connected with a set $\{x_1, \ldots, x_n\}$ of algebraic independent parameters by a system $f_i(x_1, \ldots, x_n, y_1, \ldots, y_n) = 0$ $(i = 1, 2, \ldots, n)$ of rational functions. Constructing algebraic independent parameters by Nesterenko's theorem, the authors successfully applied their method to reciprocal sums of Fibonacci numbers and determined all the algebraic relations between three q-series belonging to one of the sixteen families of q-series introduced by Ramanujan.

In this talk we apply the method mentioned above to various sets of numbers. Our algebraic independence results include the numbers $F_{2k}(q)/(q;q)_\infty^3$ from Ramanujan's Lost Notebook, where $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$ and

$$F_{2k}(q) := \sum_{n=1}^{\infty} (-1)^{n-1} (2n-1)^{2k+1} q^{n(n-1)/2}, \quad (q;q)_\infty := \prod_{n=1}^{\infty} (1 - q^n).$$

By a second application we treat the values $P(q^r), Q(q^r)$, and $R(q^r)$ of the Ramanujan functions P, Q, and R, for $q \in \overline{\mathbb{Q}}$ with $0 < |q| < 1$ and $r = 1, 2, 3, 5, 7, 10$. Finally, we consider the values given by reciprocal sums of polynomials.

Nikolay MOSHCHEVITIN (Moscow Lomonosow Univ.)

Title: *Multi-dimensional Diophantine approximations and their applications*

Abstract: In this lecture I suppose to consider the following topics.

1. **Khintchine's singular matrices.** In 1926 A. Khintchine find principal differences between one-dimensional and multi-dimensional Diophantine approximations. He discovered the phenomenon that in dimension 2 and greater there exist vectors admitting anomaly good approximations by rationals. These singular vectors (and in the case of system of linear forms in $\mathbb{R}^d$-singular matrices) may admit very unexpected Diophantine properties. One of these properties is the phenomenon of degeneracy of dimension of rational subspaces generated by the best approximations. Here we shall give a brief review of classical results and recent results by the author.
2. **Diophantine exponents.** There are different types of multi-dimensional Diophantine exponents which characterize the rate of approximation of a certain linear subspace in $\mathbb{R}^d$ by rational subspaces. A principal result witch gives lower bound for the ordinary Diophantine exponent in terms of uniform Diophantine exponent (the last may be considered as a *measure of singularity*) was obtained by V. Jarnik in 1954. Recently such a result turned out to be important in different problems in Diophantine approximations. M. Laurent showed that in two-dimensional setting Jarnik's bound cannot be improved. It is not so in dimensions greater than two. We shall discuss different definitions, give a brief review on the topic and introduce new improvements of Jarnik's inequality
3. **Kozlov's problem in uniform distribution.** In 1978 V. Kozlov posed a problem of behavior of the integral of a quasi-periodic function. This problem turned out to be connected not only to the theory of dynamical systems but mostly to Diophantine approximations. This problem was completely solved by the author in 1998. In fact the most difficulty in this problem occurs due to Khintchine's singular matrices. Here we suppose to discuss certain Diophantine aspects of this result.

Yohei TACHIYA (Keio University)

Title: *Algebraic independence of infinite products generated by Fibonacci numbers*

Abstract: Let α and β be real algebraic numbers with $|\alpha| > 1$ and $\alpha\beta = -1$. We define

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n \quad (n \geq 0).$$

If $\alpha = (1+\sqrt{5})/2$, we have $U_n = F_n$ and $V_n = L_n$ $(n \geq 0)$, where the sequences $\{F_n\}_{n\geq 0}$ and $\{L_n\}_{n\geq 0}$ are Fibonacci numbers and Lucas numbers defined, respectively, by $F_{n+2} = F_{n+1} + F_n$ $(n \geq 0)$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ $(n \geq 0)$, $L_0 = 2$, $L_1 = 1$. Let $d \geq 2$ be a fixed integer.

In this talk, we give necessary and sufficient conditions for the infinite products

$$\prod_{\substack{k=1 \\ U_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{U_{d^k}}\right) \quad (i=1,\dots,m) \quad \text{or} \quad \prod_{\substack{k=1 \\ V_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{V_{d^k}}\right) \quad (i=1,\dots,m)$$

with nonzero integers $a_1,\dots,a_m$ to be algebraically independent over $\mathbb{Q}$. As an application, we obtain the algebraic independece over $\mathbb{Q}$ of the numbers

$$\prod_{\substack{k=1 \\ F_{d^k}\neq -a_i}}^{\infty}\left(1+\frac{a_i}{F_{d^k}}\right) \quad (i=1,\dots,m)$$

for any fixed integer $d \geq 2$ and any nonzero distinct integers $a_1,\dots,a_m$. In particular, two numbers $\prod_{k=2}^{\infty}(1-1/F_{2^k})$ and $\prod_{k=1}^{\infty}(1+1/F_{2^k})$ are algebraically independent over $\mathbb{Q}$.

Takaaki Tanaka (Keio Univ.)

Title: *Algebraic independence properties related to certain infinite products and Lambert series*

Abstract: In this talk we establish algebraic independence of the values of a certain infinite product as well as its successive derivatives at algebraic numbers, using the fundamental relation between infinite products and Lambert series. The infinite product treated in this talk has a simpler form than that talked at RIMS in October 2010. The method used in the proof of algebraic independence is based on the theory of Mahler functions of several variables, whose introduction will also be included in this talk.

Ryotaro Okazaki (Doshisha Univ.)

Title: *Weber's Class Number Problem and Pohst's Lower bound for Algebraic Units*

Abstracts: Let $V_n = \mathbf{Q}(\cos(2\pi/2^{n+2}))$. Then, $V_1 = \mathbf{Q}\left(\sqrt{2}\right)$, $V_2 = \mathbf{Q}\left(\sqrt{2+\sqrt{2}}\right)$, $V_3 = \mathbf{Q}\left(\sqrt{2+\sqrt{2+\sqrt{2}}}\right)$,.... The speaker proposes to call V_n the n-th Vieté field. The ring O_n of integers of V_n is $\mathbf{Z}[2\cos(2\pi/2^{n+2})]$. Weber showed O_n is a PID (principal ideal domain) for $n = 1$, 2 and 3. Then, he asked if O_n is always a PID. Later, Bauer and Masley showed O_4 is also a PID. and Linden showed O_5 is also a PID. Recently K.Horie initiated a project for affirmative answer to Weber's problem. K.Horie, M.Horie, Fukuda, Komatsu, Morisawa made significant contribution. Fortuntately, the speaker is also involved in this project.

The class number h_n of V_n measures how far is O_n from being PID. It vanishes (i.e., $h_n = 1$) if O_n is a PID. We prove a certain proposition on the prime numbers which are coprime with every h_n ($n \geq 1$). The key idea is Pohst's lower bound on units with consideration on congruence incorporated.

Ryuta Hashimoto (Kagawa Nat. Col. Tech.)

Title: Searching discriminants with large fundamental units via continued fraction expansions

Abstract: We discuss necessary conditions for a discriminant of some quadratic order not to be large compared with its fundamental unit, in terms of continued fraction expansions.

Hajime Kaneko (Kyoto Univ.)

Title: *On the b-ary expansions of algebraic irrational numbers*

Abstract: Borel conjectured that all algebraic irrational numbers are normal in any integral base b. However, very little is known on the digits of base b expansions of algebraic irrational numbers. In this talk we consider the numbers of digit changes in the b-ary expansions of algebraic irrational numbers. I will give new lower bounds for the numbers of digit changes for certainclasses of algebraic irrational numbers. If we have time, I will also discuss the p-adic expansions of algebraic irrational numbers in the field $\mathbb{Q}_p$ of p-adic numbers.

Jonathan Sondow (New York)

Title: *Some consequences of Schanuel's conjecture*

Abstract: Assuming Schanuel's conjecture (S), we prove that if z and w are complex numbers, not 0 or 1, with z^w and w^z algebraic, then z and w are either both rational or both transcendental. A corollary is that if (S) is true, then we can find four distinct transcendental positive real numbers x, y, s, t such that the three numbers $x^y \neq y^x$ and $s^t = t^s$ are all integers. Another application is that (S) implies the transcendence of the numbers $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$, i^{i^i}, and i^{e^π}. We also prove that if (S) holds and $a^{a^z} = z$, where $a \neq 0$ is algebraic and z is irrational, then z is transcendental. This is joint work with Diego Marques. Our paper is to appear in the East-West J. of Math. A preprint is available at `http://arxiv.org/abs/1010.6216`.

Yuri V. Nesterenko (Moscow State Univ.)

Title: *On the measure of irrationality of some numbers*

Abstract: For any real irrational number α the *exponent of irrationality* $\mu(\alpha)$ is defined as the supremum of the set of all numbers κ such that inequality

$$\left|\alpha - \frac{p}{q}\right| < q^\kappa$$

has infinitely many solutions in rational numbers $\frac{p}{q}$. We know that $\mu(e) = 2$ and $\mu(\alpha) = 2$ for any algebraic irrational number α by K. Roth in 1954. Unfortunately precise values of $\mu(\alpha)$ are known only for a small set of numbers. For some classical constants we know

only upper bounds for the exponent of irrationality. For example the best up today bound

$$\mu(\pi) \leq 7,606308\ldots$$

was proved in 2008 by V.H. Salikhov. The best known inequalities

$$\mu(\zeta(2)) \leq 5,441242\ldots, \qquad \mu(\zeta(3)) \leq 5,513890\ldots,$$

were proved in 1996 and 2001 in joint papers of G. Rhin and C. Viola. In 2009 R. Marcovecchio proposed a new construction of rational approximations to logarithms of rational numbers and proved inequality

$$\mu(\ln 2) \leq 3,57455390\ldots. \tag{1}$$

In the talk we discuss the Salikhov's construction of rational approximations to π and a new construction of rational approximations to $\ln 2$ that give upper bound (1) and is simpler than Marcovecchio's one.

Yasutsugu Fujita (Nihon Univ.)

Title: *Generators for the elliptic curve $y^2 = x^3 - nx$ II*

Abstract: Let E be an elliptic curve over the rationals $\mathbb{Q}$ given by $y^2 = x^3 - nx$ with a positive integer n. Mordell's theorem asserts that the group of rational points on E is finitely generated. Our interest is in the generators for its free part. In our recent work, we gave several infinite families of n for which certain 2 points of infinite order can always be in a system of generators for the Mordell-Weil group $E(\mathbb{Q})$. In this talk, we will extend the above work and give those examples of infinite families of n for which the generators of rank 3 part for $E(\mathbb{Q})$ can be explicitly described.

Pingzhi Yuan (South China Normal Univ.)

Title: *Ko's method and its application to Diophantine equations*

Abstract: In this talk, we will present a survey on Ko's method and its application to Diophantine equations. We will give the development of the Diophantine equations of the form $AX^4 - BY^2 = \pm 1, 2, 4$. Now most equations of the form $AX^4 - BY^2 = \pm 1, 2, 4$ have been completely solved by Ko's method and the method of Thue-Siegel. We can also improve the above two methods to solve other interesting equations, for example, we proved that the diophantine equation $x^2 - (1 + a^2)y^4 = -2a, a > 0$ has at most three positive integer solutions (x, y); and we proved that: Let $a > 1$ and $D > 1$ be positive integers, and let p be an odd prime which does not divide aD. If $(a, D, p) \neq (1, 2, 5), (1, 4, 5), (2, 7, 3), (2, 7, 9)$, then the diophantine equation $ax^2 + D^m = p^n$ has at most two positive integer solutions (x, m, n). Moreover, the diophantine equations $x^2 + 2^m = 5^n, x^2 + 4^m = 5^n, 2x^2 + 7^m = 3^n$ and $2x^2 + 7^m = 9^n$ have precisely three positive integer solutions (x, m, n).

Takafumi MIYAZAKI (Tokyo Metropolitan Univ.)

Title: *Exceptional cases of Terai's conjecture on Diophantine equations*
Abstract: Let p, q, r be positive integers greater than 1, and let a, b, c be relatively prime positive integers such that $a^p + b^q = c^r$. Terai's conjecture states that if (a, b, c) is not any of the following cases (taking $a < b$):

$$1^i + 2^3 = 3^2;\ i \geq 2, \quad 2^5 + 7^2 = 3^4, \quad 2^p + (2^{p-2} - 1)^2 = (2^{p-2} + 1)^2;\ p \geq 4$$

(we call these cases *exceptional cases*), then the only solution of the equation

$$a^x + b^y = c^z, \quad x, y, z \in \mathbb{N}$$

is $(x, y, z) = (p, q, r)$. In this talk, we consider the case where $q = r = 2$ and give some results related to exceptional cases.

Tomohiro YAMADA (Kyoto Univ.)

Title: *On diophantine equation $x^2 = p^g + p^f + p^e + 1$*
Abstract: We show that the equation $x^2 = p^g + p^f + p^e + 1$ has only finitely many solutions in integers x, e, f, g for each prime p under the condition e is even or both of f, g are even. We give an explicit bound for such solutions x, e, f, g.

Shigeki AKIYAMA (Niigata Univ.)

Title: *On the spectrum of* $\{-1, 0, 1\}$ *polynomials*
Abstract: Given $q > 1$, consider the set

$$Y(q) = \left\{ \sum_{i=0}^{n} a_i q^i \,\middle|\, a_i \in \{-1, 0, 1\},\ n = 0, 1, \dots \right\}$$

which is sometimes called the spectrum of q. Originated by P. Erdős and his collaborators, many authors studied its topology in relation to the expansion in non-integer base q, Diophantine property of $Y(q)$, and characterization of singular Bernoulli convolution. Our main question is for which q, the spectrum $Y(q)$ is dense, discrete, uniformly discrete in $\mathbb{R}$. We review the situation and explain our recent best possible result on discreteness:

$$Y(q) \text{ is closed and discrete if and only if } q \text{ is a Pisot number or } q \geq 2$$

obtained together with V. Komornik. This result gives many consequences for denseness and uniformly discreteness as well, but they are not best possible and leave us interesting open questions.

Takumi NODA (Nihon Univ.)

Title: *On generalized Lipschitz-type formulae and applications II*
Abstract: Let $s = \sigma + it$ be a complex variable and let $H^+ = \{z \in \mathbb{C} \mid 0 < \arg(z) < \pi\}$

and $H^- = \{z \in \mathbb{C} \mid -\pi < \arg(z) < 0\}$ be the complex half planes. The Lipschitz formula is known as the following equality

$$\sum_{m=-\infty}^{\infty} (z+m)^{-k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} \exp(2\pi i n z) \qquad (z \in H^+),$$

for positive integer k, and is also known as one of the key ingredients for obtaining the Fourier expansion of the Eisenstein series.

The purpose of this talk is to describe a double analogue of the Lipschitz formula via the twisted Mellin-Barnes integral transformation. We also define a class of double Eisenstein series on $H^+ \times H^-$, which includes the real-analytic Eisenstein series for $SL(2, \mathbb{Z})$, and give the transformation formula by using the generalized Lipschitz-type formula.

Masanori KATSURADA (Keio Univ.)

Title: *Shintani zeta-functions of several variables and Lauricella hypergeometric functions*

Abstract: Let $n \geq 1$ be an integer, s and s_j $(j = 1, \dots, n)$ complex variables, and a_j, λ and λ_j real numbers with $a_j > 0$ $(j = 1, 2)$. The vectorial notation $\boldsymbol{x} = (x_1, \dots, x_m)$ with the abbreviation $\langle \boldsymbol{x} \rangle = x_1 + \cdots + x_n$ for any integer $m \geq 1$ and any complex numbers x_i $(i = 1, \dots, m)$ is used hereafter. We further write $e(\lambda) = e^{2\pi i \lambda}$, and let z_j be complex parameters with $|\arg z_j| < \pi$ $(j = 1, \dots, n)$. The main object of this talk is the Shintani zeta-function $\widetilde{\phi}_n$ of n-variables $\boldsymbol{s} = (s_1, \dots, s_n)$, in the form

$$\widetilde{\phi}_n(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\lambda}; \boldsymbol{z}) = \sum_{l_1, l_2 = 0}^{\infty} e(\lambda_1 l_1 + \lambda_2 l_2) \prod_{j=1}^{n} \{a_1 + l_1 + (a_2 + l_2) z_j\}^{-s_j}, \tag{1}$$

which converges absolutely for $\mathrm{Re}\langle \boldsymbol{s} \rangle > 2$. We write $\mathbf{1} = (1, \dots, 1)$. Then the 'diagonal' case $\boldsymbol{s} = s\mathbf{1}$ of (1) is reduced to the one variable (double) zeta-function $\phi_n(s, \boldsymbol{a}, \boldsymbol{\lambda}; \boldsymbol{z}) = \widetilde{\phi}_n(s\mathbf{1}, \boldsymbol{a}, \boldsymbol{\lambda}; \boldsymbol{z})$, whose (full) multiple form was first introduced and studied by T. Shintani in 1976.

It is the principal aim of this talk to present asymptotic aspects of $\widetilde{\phi}_n(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\lambda}; \boldsymbol{z})$ when z_j $(j = 1, \dots, n)$ are both small and large; this leads us to show complete asymptotic expansions of $\widetilde{\phi}_n(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{\lambda}; \boldsymbol{z})$ in the ascending order of z_n as $z_n \to 0$, and also in the descending order of z_n as $z_n \to \infty$, both through the sectorial region $|\arg z_n - \theta_0| < \pi/2$ for any angle θ_0 fixed with $|\theta_0| < \pi/2$, while other z_j's are in the same sector upon satisfying the conditions $z_j \asymp z_n$ $(j = 1, \dots, n-1)$. Here $A \asymp B$ means that both $A \ll B$ and $A \gg B$ hold. Several applications of our main formulae will further be given.