# *Regular Substitution Systems and Tilings in the Hyperbolic Plane*

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An advertisement for The Tiling Listserve

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In the hyperbolic plane (as well as in other settings) we must distinguish between *weakly* and *strongly* aperiodic sets of tiles.

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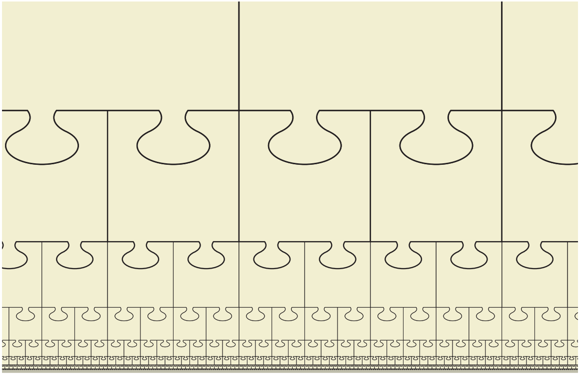

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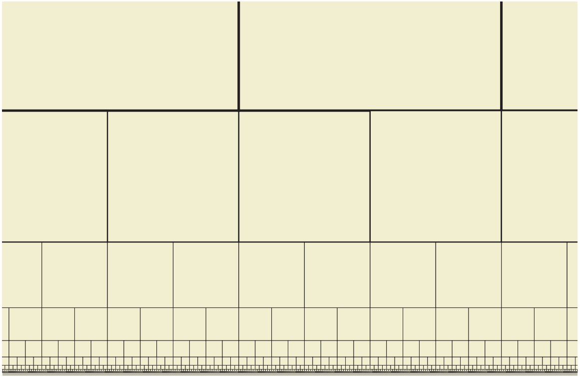

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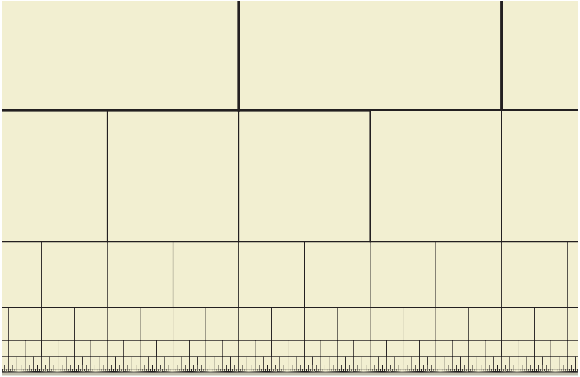
For example, ca. 1977, Penrose noted there are *weakly* aperiodic sets of tiles in the hyperbolic plane.



This tile admits tilings with infinite cyclic symmetry, but none with finite fundamental domain. The tile is *weakly* aperiodic.

In the hyperbolic plane, a set of tiles is *weakly* aperiodic if it does admit tilings, but admits no tiling with a compact fundamental domain. On the other hand, it may admit tilings with an infinite cyclic symmetry.

A set of tiles in *strongly* aperiodic if it does admit tilings, but admits no tiling with even an infinite cyclic symmetry.

Though the Domino Problem is tied to weak aperiodicity, somehow strong aperiodicity seems to be a more significant condition.

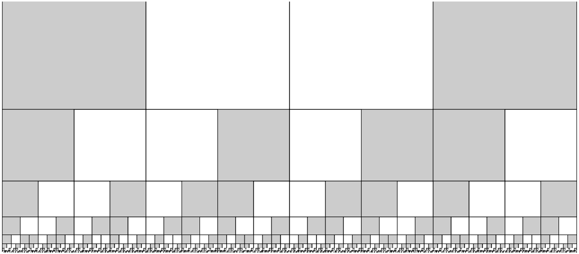
And indeed, as we will see, weakly aperiodic sets of tiles are commonplace.

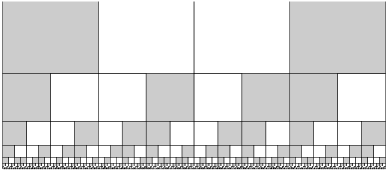
Given substitution system on letters, we can construct, in fact, a weakly aperiodic set of tiles, based on the induced substitution tiling of the line:

For example, based on the Morse-Thue substitution system

$$0 \to 01, \qquad 1 \to 10$$

the horocyclic rows in the tiling correspond to bi-infinite words in the system. A row is above another exactly when the corresponding words are related by substitution.



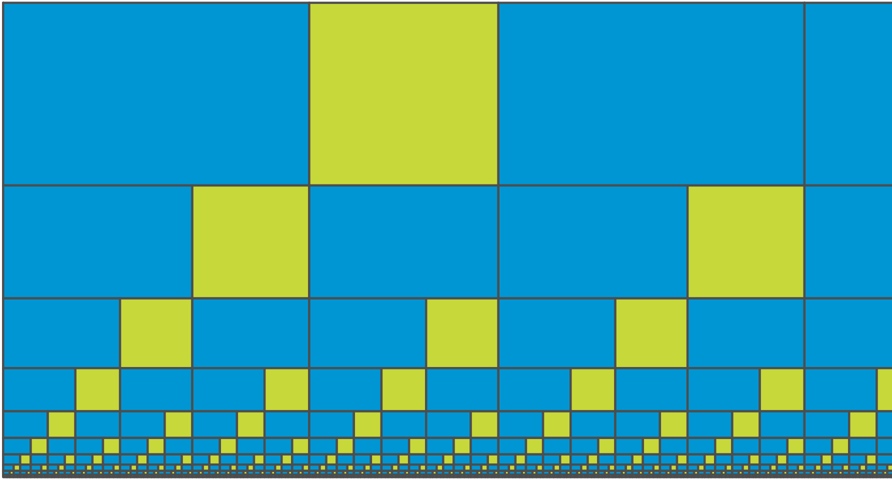


The tilings admitted by these tiles exactly correspond to the orbits of bi-infinite strings in the substitution system.

Though these tiles do not admit a tiling with compact fundamental domain, they do admit a tiling invariant under an infinite cyclic symmetry, since there are periodic orbits in the substitution system.

This pair of tiles is only **weakly aperiodic**.

We see that every primitive substitution system corresponds to a weakly aperiodic set of tiles in $H^2$ in precisely this manner.



$0 \longrightarrow 1$

$1 \longrightarrow 10$

$\frac{\sqrt{5}+1}{1}$

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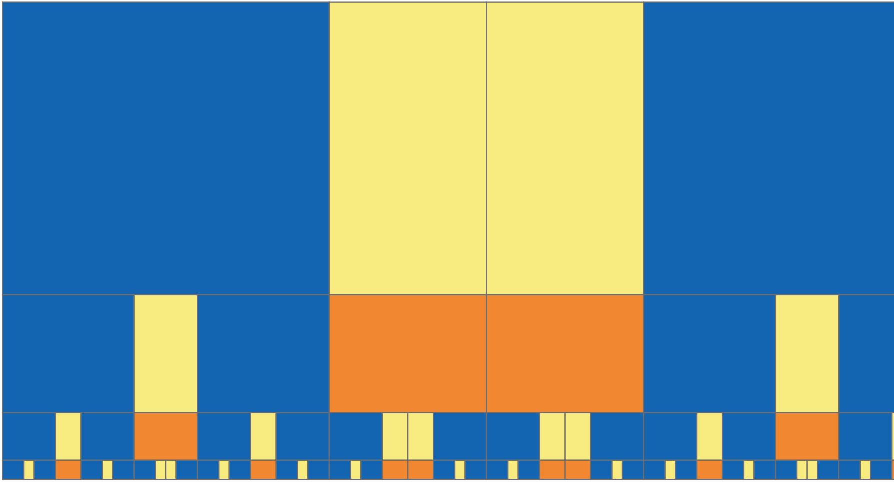

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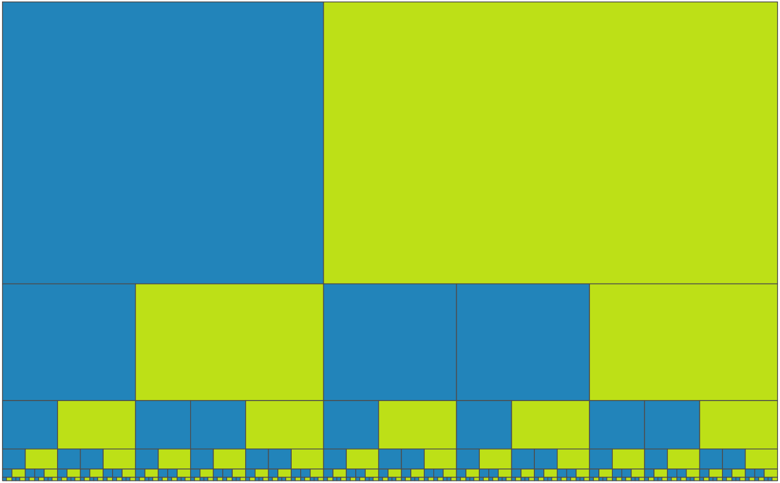

$0 \longrightarrow 110$

$1 \longrightarrow 10$

$1+\sqrt{2}$

Note that the well-defined expansion rate of the substitution (arising as the lead eigenvalue of the substitution matrix) induces a particular curvature on the tiles.

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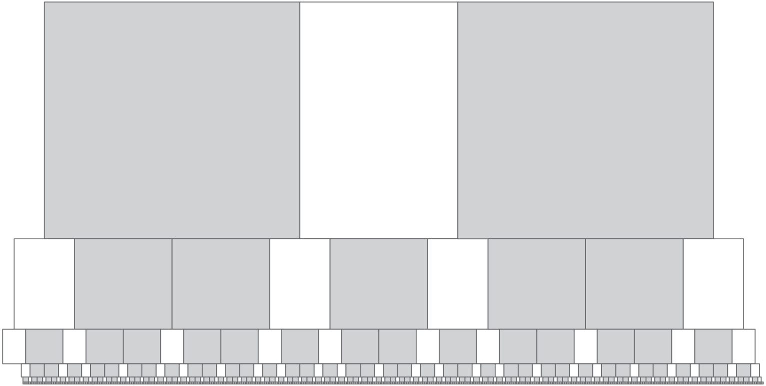
However, countably many of these orbits are periodic, and countably many of the tilings have an infinite cyclic symmetry.

Consequently such tiles are *weakly* aperiodic.

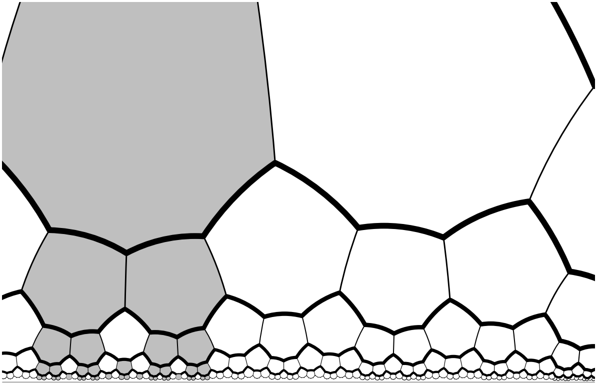
**An amusing phenomenon:** If we horizontally shift the tiles of the "fibonacci-squared" substitution:



$0 \longrightarrow 10$
$1 \longrightarrow 110$

we obtain *exactly* a combinatorially regular tiling of heptagons meeting three-to-a-vertex.

And so too the regular $\{7, 3\}$ tiling can be shelled into roughly horocyclic layers, arranged by the fibonacci-squared substitution system:



(Interestingly, there are uncountably many distinct ways to do this, corresponding to the uncountable set of orbits under in the substitution system.

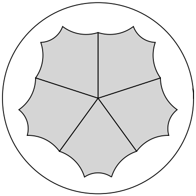
The countably many periodic orbits correspond to the countably many conjugacy classes of infinite cyclic subgroups of the symmetry group of the regular tiling.

Does the set of all orbits correspond to the set of points at infinity, modulo the symmetry group?)

This basic idea is quite general; a wide range of tilings can be modeled by substitution sequences. Let's sketch:

## Theorem

*(Poincaré) For any $p, q \geq 3$ with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ there is a tiling of $\mathbf{H}^2$ by regular p-gons meeting q-to-a-vertex.*

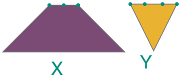


At least locally, the construction is more or less trivial. With simple trigonometry, one can construct a regular $p$-gon with vertex angles $2\pi/q$,

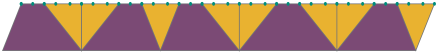
and thus construct an arrangement of $q$ regular $p$-gons surrounding a single vertex. The question is, how to extend this to a global tiling. *Is it possible something might go wrong?*

# How to construct a tiling in $\mathbf{H}^2$

{5,5}

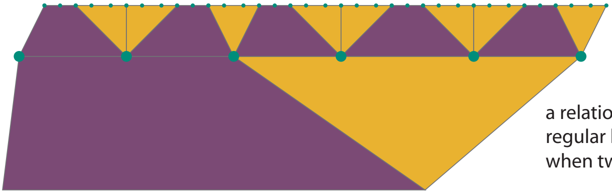



letters describe local configurations



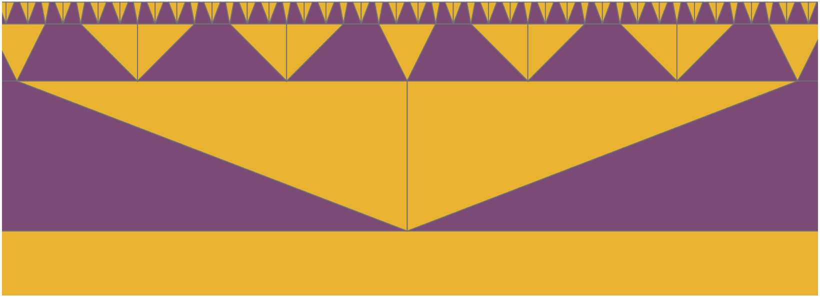
X YYXYXYYXYYXY

words describe configurations along a curve
locally sensible configurations are described
by words in a regular language

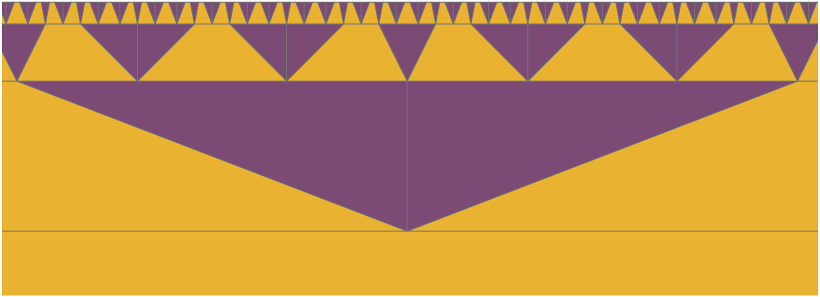


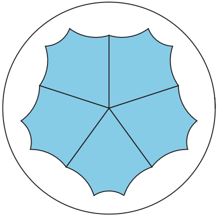
a relation on words in this
regular language describes
when two strips can fit together.

XY -> XYYXYXYYXYYXY



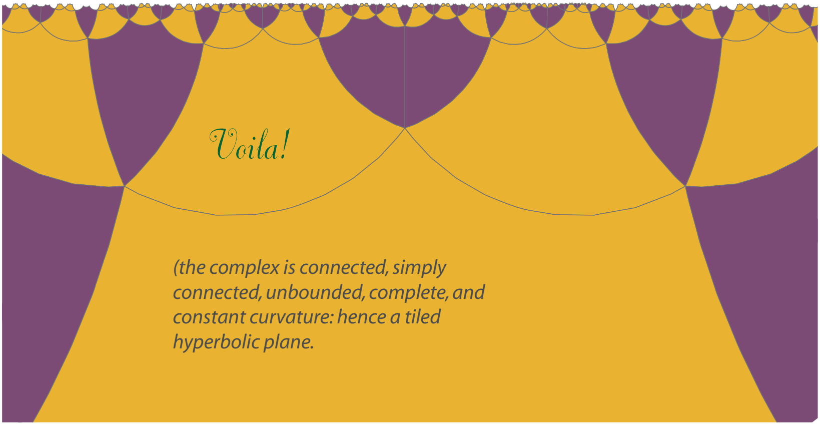
a bi-infinite orbit under this relation, on the bi-infinite words in the language corresponds to an abstract complex with the correct local combinatorics.



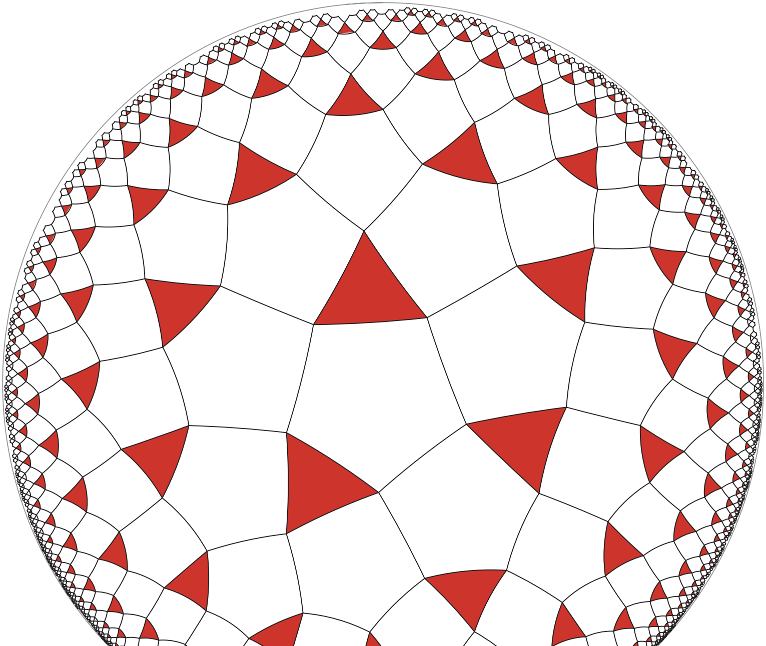


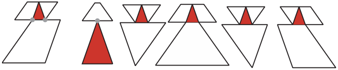
We began with a local geometric realization of the desired combinatorial structure.

We simply use this to chart a geometry onto the complex.

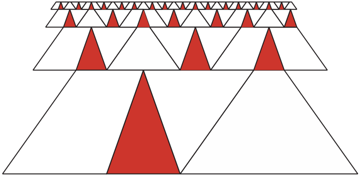

Voila!
(the complex is connected, simply connected, unbounded, complete, and constant curvature: hence a tiled hyperbolic plane.

This basic technique is very powerful. For example, how can we discuss the existence of this non-periodic archimedean tiling?



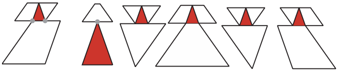


the alphabet, rules and language



a bit of the complex

These rules, and the existence of orbits under the substitution, ensure that the tiling exists.



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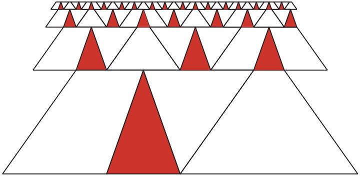

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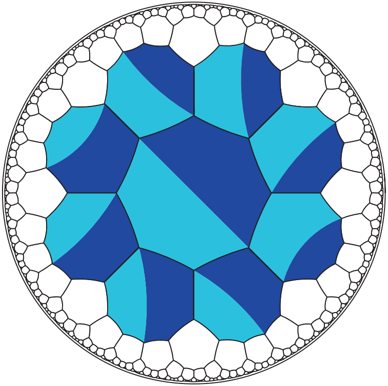
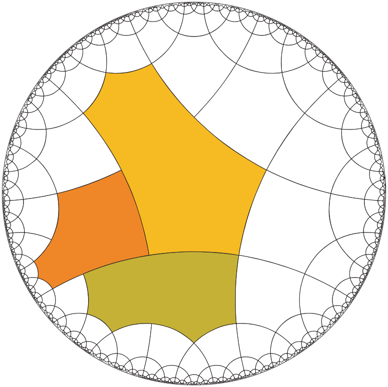
Danzer posed a curious question: Can the octagons in this tiling be halved, consistently, to obtain a tiling by pentagons meeting in fours?



Can the pentagons in this tiling be fused to obtain a tiling by octagons meeting in threes?



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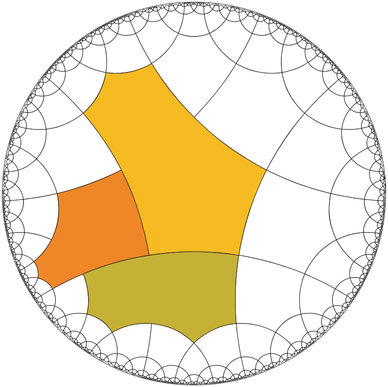
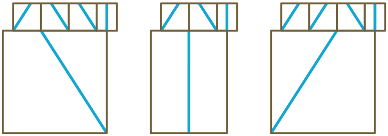
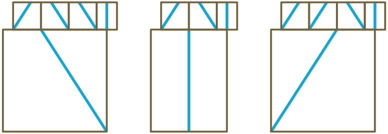


The difficulty is that a *local* method can be found, but how *can this be extended globally?*



The solution is to find rules that capture the desired combinatorics.



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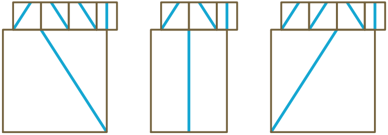
By painting a different geometry on the complex, we obtain the different desired tilings.



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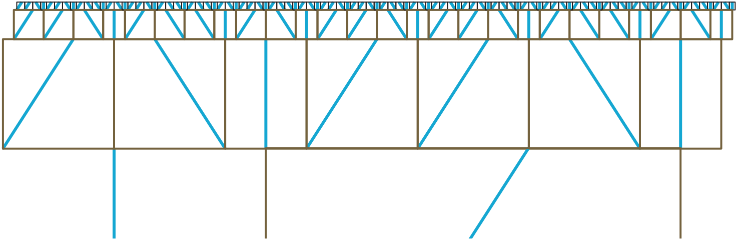

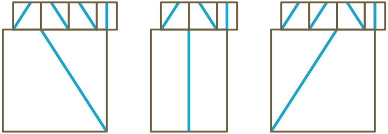

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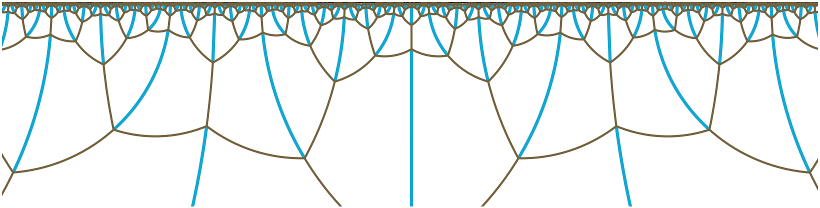

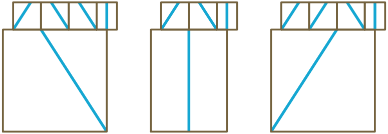

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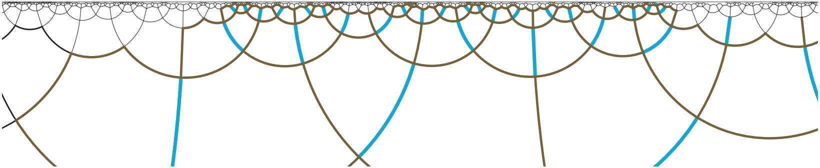
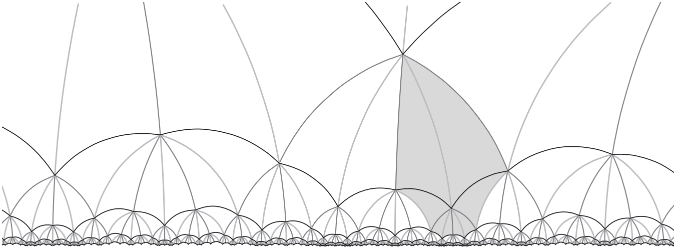


Another application: Triangle Tilings. **Thm** *Let $T$ be a triangle in the hyperbolic plane with vertex angles $a, b, c$. Suppose there exist unique integers $r, s, t \geq 0$ with $ra + sb + tc = 2\pi$. Then $T$ admits a tiling iff $r \equiv s \equiv t \bmod 2$*

This covers some pretty strange triangles! For example, no triangle satisfying the hypotheses of the theorem admits a tiling with a co-compact symmetry. Those that admit tilings do in fact admit tilings with an infinite cyclic symmetry however, and are thus *weakly aperiodic*!



In particular, among all possible triangles (in $S^2, E^2$ and $H^2$) it is easy to see a measure 1 set do not admit any tiling.

This theorem states that among the rest, *a measure 1 set of tiling triangles are weakly aperiodic!*

(The classical Poincaré triangles are only a measure 0 set all together.)

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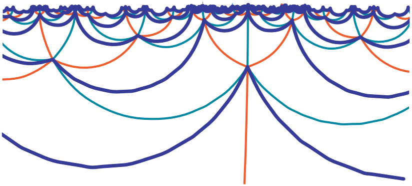

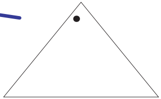

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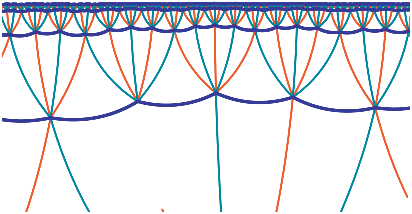

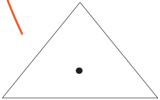

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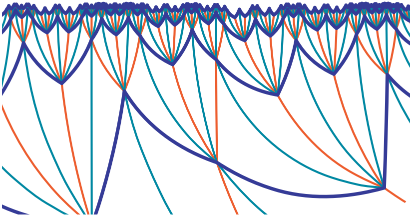

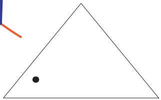

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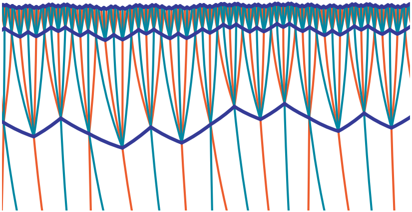
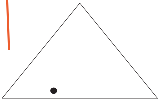




**More generally tilings are far more subtle**

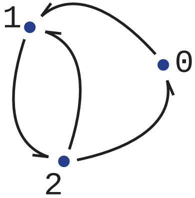
In the constructions we just gave, we were able to find a classical symbolic substitution system within our construction.

Thus we were able to construct orbits, and indeed periodic orbits.

But in general, there may be no orbits, or there may be orbits but no periodic orbits.

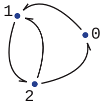
Consider the following example, which more closely models the generic nature of tilings:

Consider a language $L$ defined as paths in this graph:




Now take as rules

$$0 \mapsto 12$$
$$1 \mapsto 12 \qquad 1 \mapsto 21$$
$$2 \mapsto 01 \qquad 2 \mapsto 20$$




$$\begin{array}{ll} & 0 \mapsto 12 \\ 1 \mapsto 12 & 1 \mapsto 21 \\ 2 \mapsto 01 & 2 \mapsto 20 \end{array}$$

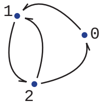
For words $w, v \in L$ write $w \mapsto v$ iff there is *some choice* of replacements of the letters in $w$ yielding $v$. Note that a given word may map to no, one, or several other words.

For example:

$012120 \mapsto$

$0120 \mapsto$

$1212 \mapsto$

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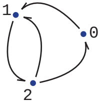


| | |
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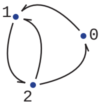


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$0120 \mapsto$ nothing

$1212 \mapsto$ 12012120, 21201201

As before this relation extends to a relation on the infinite strings $L^\infty$. *Is there an orbit?*

As before this relation extends to a relation on the infinite strings $L^\infty$. *Is there an orbit?*

(The restriction to a regular language reflects that tiles are subject to local constraints: what fits with what. The possibility of choices in the substitution reflects that more than one tile might fit with another.)

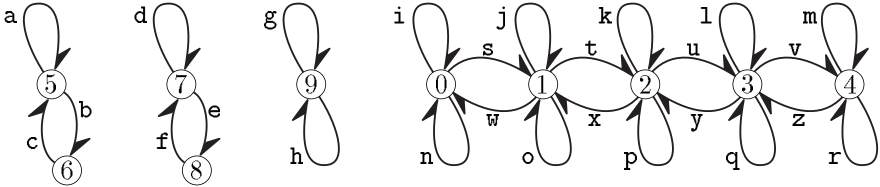
A "regular substitution system" $(A, L, R)$ is specified by:

a regular language $L$ on an alphabet $A$; and a relation $R$, satisfying certain axioms, on words in this language. The relation extends to a relation on $L^\infty$.

In general, ask, given $(A, L, R)$ are there orbits in $L^\infty$?

Are there periodic orbits?

Quite unlike the classical symbolic substitution dynamical case, this is quite subtle. Indeed, here is an example of an $(A, L, R)$ for which there *is* an orbit, yet there is *no* periodic orbit:

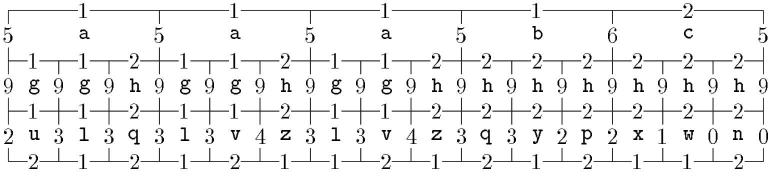



a, d $\longrightarrow$ ggh  b, c $\longrightarrow$ hhh  e, f $\longrightarrow$ ggg

g $\longrightarrow$ i, j, k, l, m, s, t, u, v  i, j, k, l, m, w, x, y, z $\longrightarrow$ a, b, e

h $\longrightarrow$ n, o, p, q, r, w, x, y, z  n, o, p, q, r, s, t, u, v $\longrightarrow$ c, d, f

(This example can also be interpreted as a **strongly aperiodic** set of tiles in $\mathbf{H}^2$).



**Decidability and regular production systems** The existence of tilings admitted by a given set of tiles is equivalent to the existence of orbits in a corresponding regular production system. Is this existence decidable? Is there an algorithmic way to answer whether a given regular production system has orbits.

Berger's celebrated result (1966), that the "Domino Problem" is undecidable in the Euclidean plane, can be interpreted as showing that in fact whether a regular production system has orbits is undecidable, at least for systems with linear growth rates.

The Domino Problem has now been shown to be undecidable in the hyperbolic plane (Margenstern 2007) (Kari, 2007) These production systems are thus quite subtle– we cannot decide if they admit orbits, nor periodic orbits.

**Curvature and Production Systems** Let us turn the construction around, and beginning with an arbitrary production system—which we assume has orbits—ask whether the system corresponds to a tiling with constant (or approximately constant) curvature.

As we saw earlier, symbolic substitution systems have a well-defined asymptotic growth rate. This rate in turn determines at least something of the geometry of the corresponding tiles.

Roughly, if growth is exponential, we can endow the system with a hyperbolic geometry; if linear, the system is Euclidean; etc. If word length is wildly erratic, no consistent geometry can be placed on the system.

### Conjecture

*Given a regular production system which admits orbits, it is undecidable whether there is an asymptotic rate of growth.*

A well-known problem can be interpreted as an example of this phenomenon.

**The Kolakoski sequence** consists of 1's and 2's, occuring singly or in pairs, and has the remarkable property that it "reads" its own structure:

$$1\ 22\ 11\ 2\ 1\ 22\ 1\ 22\ 11\ \ldots$$

A well-known problem can be interpreted as an example of this phenomenon.

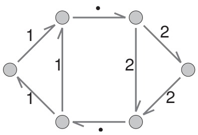
**The Kolakoski sequence** consists of 1's and 2's, occuring singly or in pairs, and has the remarkable property that it "reads" its own structure:

1 22 11 2 1 22 1 22 11 ...

1 2 2 1 1 2 1 2 2

It is an well-known open question whether, in the limit, there are as many 1's as 2's.

The construction of the Kolakoski sequence is captured by a particular regular production system:




$$1 \to 1. \qquad 2 \to 11.$$

$$1 \to 2. \qquad 2 \to 22.$$

$$. \to$$

Q: Does this system have an asymptotic rate of growth, and if so is this rate 3/2?