# RECURSIVELY RENEWABLE WORDS AND CODING OF IRRATIONAL ROTATIONS

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Abstract. A word generated by coding of irrational rotation with respect to a general decomposition of the unit interval is shown to have an inverse limit structure directed by substitutions. We also characterize primitive substitutive rotation words, as those having quadratic parameters.

## 1. Definitions and the results

Let $\mathcal{A} = \{0, 1, \dots, m-1\}$ be a finite set of letters and $\mathcal{A}^*$ be the monoid over $\mathcal{A}$ generated by concatenation, having the identity element $\lambda$, the empty word. The set of right infinite words over $\mathcal{A}$ is denoted by $\mathcal{A}^{\mathbb{N}}$.

A sturmian word $z$ is an element of $\mathcal{A}^{\mathbb{N}}$ characterized by the property that $p_z(n) = n+1$, where $p_z(n)$ is the number of factors (i.e. subwords) of length $n$ appears in $z$. The function $p_z(n)$ is called the complexity of $z$. Since $p_z(1) = 2$, we have $\mathcal{A} = \{0, 1\}$. The sturmian word is known to have the lowest complexity among aperiodic words. The aperiodicity implies that exactly one of $\{00, 11\}$ appears in $z$. Let us assume that 11 is forbidden in $z$. Then the sturmian word $z = z_0 z_1 \dots \in \{0, 1\}$ with $z_0 = 0$ allows a decomposition into a word over $\mathcal{B} = \{0, 01\}$. An important fact is that this new word over $\mathcal{B}$ is again a sturmian word. This property is effectively used to recode sturmian words by the continued fraction algorithm (see Chap. 6 in [18]). We wish to generalize this combinatorial property.

Let us come back to a general $\mathcal{A} = \{0, 1, \dots, m-1\}$. An element $z = z_0 z_1 \dots \in \mathcal{A}^{\mathbb{N}}$ is *$k$-renewable* if there is a finite set $\mathcal{B} \subset \mathcal{A}^*$ with $^{\#}\mathcal{B} \le k$ and $\mathcal{B} \not\subset \mathcal{A}$ such that $z$ is decomposed into an infinite word over $\mathcal{B}$. For a given $k$, if the element $z \in \mathcal{A}^{\mathbb{N}}$ allows infinitely many times this decomposition into $k$-blocks, then $z$ is called *recursively $k$-renewable*. To be more precise, $z = z_0 z_1 \dots$ is recursively $k$-renewable when there is a sequence of finite sets $\mathcal{B}_i$ $(i = 0, 1, \dots)$ with $^{\#}\mathcal{B}_i \le k$, $\mathcal{B}_{i+1} \subset \mathcal{B}_i^*$, $\mathcal{B}_{i+1} \not\subset \mathcal{B}_i$, $\mathcal{B}_0 = \mathcal{A}$ and $z^{(i)} = z_0^{(i)} z_1^{(i)} \dots \in \mathcal{B}_i^{\mathbb{N}}$ is $k$-renewable by $\mathcal{B}_{i+1}$ and decomposed into $z^{(i+1)} = z_0^{(i+1)} z_1^{(i+1)} \dots \in \mathcal{B}_{i+1}^{\mathbb{N}}$ and the length of each $z_j^{(i+1)}$ as a word in $\mathcal{A}^*$ diverges[1] as $i \to \infty$. Here we put $z_i^{(0)} = z_i$ for $i = 0, 1, \dots$. We also say that $z \in \mathcal{A}^{\mathbb{N}}$ is *recursively renewable* if it is recursively $k$-renewable with some $k$.

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[1] We must assume this divergence to remove trivial cases (a remark due to J. Cassaigne).

**Example 1.** *A purely periodic word $vvv\dots$ with $v \in \mathcal{A}^*$ is recursively* 1*-renewable by taking $\mathcal{B}_i = \{v^{2^i}\}$. An eventually periodic word $uvvv\dots$ with $u, v \in \mathcal{A}^*$ is recursively* 2*-renewable by $\mathcal{B}_i = \{uv^{2^{i-1}}, v^{2^i}\}$.*

**Example 2.** *The sturmian word is recursively* 2*-renewable.*

Let $\mathcal{C}$ be a non empty finite set. A *morphism* $\sigma$ is a monoid homomorphism from $\mathcal{A}^*$ to $\mathcal{C}^*$ that $\sigma(a) \neq \lambda$ for each $a \in \mathcal{A}$. Then $\sigma$ naturally extends to a map from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{C}^{\mathbb{N}}$. A morphism $\sigma$ is called *letter to letter*, if $\sigma(a) \in \mathcal{C}$ for each $a \in \mathcal{A}$. A *substitution* is a morphism from $\mathcal{A}^*$ to itself.

**Example 3.** *Let $\sigma$ be a morphism from $\mathcal{A}^*$ to $\mathcal{C}^*$. The image by $\sigma$ of a recursively $k$-renewable word in $\mathcal{A}^{\mathbb{N}}$ is recursively $k$-renewable in $\mathcal{C}^{\mathbb{N}}$.*

**Example 4.** *Let $\sigma$ be a substitution on $\mathcal{A}^*$. If $\sigma(0) = 0w$ with $w \neq \lambda$, then there is a unique fix point $z$ of $\sigma$ in $\mathcal{A}^{\mathbb{N}}$ which begins with the letter* 0*. The word $z$ is successively approximated by $\sigma^n(0)$ $(n = 1, 2, \dots)$ and we write $z = \lim_n \sigma^n(0)$. The fix point $z$ is recursively $m$-renewable by $\mathcal{B}_i = \{\sigma^i(0), \dots, \sigma^i(m-1)\}$.*

In other words, $z$ is recursively $k$-renewable if there is a sequence $\{\phi_i\}_{i=1,2,\dots}$ of substitutions on $\{0, 1, \dots, k-1\}$ that

$$z \xleftarrow{\phi_1} z_2 \xleftarrow{\phi_2} z_3 \xleftarrow{\phi_3} z_4 \xleftarrow{\phi_4} \dots$$

with $z_i \in \{0, 1, \dots, k-1\}^{\mathbb{N}}$ and the length of $\phi_1\phi_2 \dots \phi_i(a)$ diverges for each letter $a$ as $i \to \infty$, i.e., $z$ lies in $\varprojlim_{\phi_i} \{0, 1, \dots, k-1\}^{\mathbb{N}}$, the inverse limit directed by $\{\phi_i\}$.

We recall the definition of general rotation words. Take $\xi \in [0, 1) \setminus \mathbb{Q}$ and $\mu \in [0, 1)$. Start with a decomposition of the unit interval

$$I = [0, 1) = \bigcup_{i=0}^{k-1} [\omega_i, \omega_{i+1}) \tag{1}$$

with $0 = \omega_0 < \omega_1 < \dots < \omega_{k-1} < \omega_k = 1$ and put $I_i = [\omega_i, \omega_{i+1})$. We identify $[0, 1)$ with the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and define $J : \mathbb{T} \to \{0, 1, \dots, k-1\}$ by $x \in I_{J(x)}$ for $x \in \mathbb{T}$. Then the *general rotation word* of an angle $\xi$ and an initial value $\mu$ with respect to $k$-block decomposition (1) is defined by

$$J(\mu)J(\mu + \xi)J(\mu + 2\xi) \dots \in \{0, 1, \dots, k-1\}^{\mathbb{N}}.$$

The classical rotation word comes from the decomposition $[0, 1) = [0, 1-\xi) \cup [1-\xi, 1)$. It is well known (c.f [30], [10], [28], [18]) that the set of all classical rotation word coincides with that of all sturmian words.[2] For classification of words of complexity $2n$, Rote [33] used the generalized rotation word with respect to 2-blocks and also gave a combinatorial characterization of the words with respect to $[0, 1/2) \cup [1/2, 1)$. Further connection between general rotation words and sturmian words had been studied. Didier [12] characterized the general rotation words in terms of sturmian words and cellular automata. Berstel and Vuillon [3] showed a way to recode generalized rotation words with respect to $k$-blocks into $k$-tuples of sturmian words.

In this paper, we first prove the following theorem.

[2]Precisely, we need to consider another decomposition $(0, 1] = (0, 1-\xi] \cup (1-\xi, 1]$ in order to show the equivalence to sturmian words.

**Theorem 1.** *A general rotation word of an angle $\xi$ and an initial value $\mu$ with respect to $k$-block decomposition* (1) *is recursively* $(k+1)$*-renewable.*

There are recursively renewable words which can not be a general rotation word. For examples, take a fix point $z$ of the substitution

$$\tau(0) = 001, \tau(1) = 111$$

introduced by Rote [33] as a concrete word with $p_z(n) = 2n$. Example 4 says that $z$ is recursively 2-renewable but $z$ contains factors $1^n$ for all $n \geq 1$, which is impossible for a general rotation word. Another example is the fix point $g$ of the Rauzy substitution:

$$\sigma(0) = 01, \quad \sigma(1) = 02, \quad \sigma(2) = 0,$$

having its complexity $p_g(n) = 2n + 1$. Then $g$ is recursively 3-renewable but not a general rotation word. Indeed, $g$ has arbitrary large special factors $w$ that $w0, w1, w2$ are also the factors of $g$, but general rotation words can not have this property. The maximal pattern complexity also tells apart that $g$ can not be a general rotation word (c.f. [23]). It is known that $g$ is a coding of 6-interval exchange of $\mathbb{T}$ and moreover a natural coding of a rotation on $\mathbb{T}^2$ (c.f [2], [32]). It remains a problem to characterize generalized rotation words among recursively renewable words. In addition, authors got to know a relevant result [29] with Theorem 1 after submission of this paper.

An element $z \in \mathcal{A}^{\mathbb{N}}$ is *primitive substitutive* if it is an image of a morphism of a fixed point of a primitive substitution.[3] Among recursively renewable words viewed as elements of the inverse limit, primitive substitutive words correspond to eventually periodic sequences $\{\phi_i\}_{i=1,2,\ldots}$ of substitutions. Durand [14] and Holton-Zamboni [19] independently[4] gave a combinatorial characterization of primitive substitutive words using return words. With the help of their result and the idea of the proof of Theorem 1, we can characterize primitive substitutive rotation words;

**Theorem 2.** *A general rotation word of an angle $\xi$ and an initial value $\mu$ with respect to the decomposition* (1) *is primitive substitutive if and only if $\xi$ is quadratic irrational, $\mu \in \mathbb{Q}(\xi)$ and $\omega_i \in \mathbb{Q}(\xi)$ for all $i$.*

Note that the last condition '$\mu \in \mathbb{Q}(\xi)$ and $\omega_i \in \mathbb{Q}(\xi)$ for all $i$' is equivalent to '$\omega_i - \mu \in \mathbb{Q}(\xi)$ for all $i$'.

Theorem 2 generalizes Theorem 7.8 of [1], the same result for $[0, \omega_1) \cup [\omega_1, 1)$ under a condition $\omega_1 \notin \mathbb{Z}+\xi\mathbb{Z}$, and Proposition 2.11 of [7] which treated all sturmian case, i.e., $[0, 1-\xi) \cup [1-\xi, 1)$. Theorem 2 may also be expected from number of quadratic type results on characterization of sturmian words fixed by substitutions (c.f. [11], [26], [22], [37] and [31]). The proof of Theorem 2 seems more number theoretical than those in [1] and [7], and §6 - §10 are devoted to it. A basic idea is to show directly that return words with respect to a long prefix give a coding of a certain three interval exchange. Ostrowski's numeration system in §9 is unconventionally used to control induced discontinuities and to deduce unique ergodicity of three interval exchanges in §10.

[3] It is equivalent to being an image of a letter to letter morphism of a fixed point of a primitive substitution. See Proposition 3.1 in [14].

[4] From p.13 and p.34 of [13].

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## 2. Negative continued fraction and induced rotations

Let $\lfloor x \rfloor$ be the maximum integer not greater than $x$, and $\lceil x \rceil = -\lfloor -x \rfloor$, i.e., the minimum integer not less than $x$. Define a map $S : (0,1) \to (0,1)$ by $S(x) = \lceil 1/x \rceil - 1/x$ and set $a_n = \lceil 1/S^{n-1}(x) \rceil$ and $x_n = S^n(x)$ for $x \in (0,1) \setminus \mathbb{Q}$. Then we have

$$x = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots - \cfrac{1}{a_n - x_n}}}} \tag{2}$$

with $a_n \geq 2$ and also an infinite continued fraction:

$$x = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \ddots}}}. \tag{3}$$

Both of them are called the *negative continued fraction* of $x$. There are infinitely many $n$'s such that $a_n > 2$ in the infinite continued fraction. Let $\xi_0 = 1$ and $\xi_1 = \xi$. We define $\xi_{n+1} \in [0, \xi_n)$ inductively by:

$$\xi_{n+1} = a_n \xi_n - \xi_{n-1}$$

with a positive integer $a_n$. The choice of $a_n$ is unique since $\xi_{n+1} \in [0, \xi_n)$ and $\xi \notin \mathbb{Q}$ implies $\xi_n \neq 0$. We can easily show

$$\xi = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_n - \cfrac{\xi_{n+1}}{\xi_n}}}}} \tag{4}$$

and $a_n \geq 2$. The expansion (2) of $\xi$ clearly coincides with this expansion with $x_n = S^n(\xi) = \xi_{n+1}/\xi_n$. Define integer sequences by

$$\begin{aligned} P_{n+1} &= a_n P_n - P_{n-1} \\ Q_{n+1} &= a_n Q_n - Q_{n-1} \end{aligned} \tag{5}$$

for $n \geq 1$ with initial values $(P_0, P_1, Q_0, Q_1) = (-1, 0, 0, 1)$. A useful matrix representation:

$$\begin{pmatrix} -P_n & P_{n+1} \\ -Q_n & Q_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & a_n \end{pmatrix}$$

allows us to show

$$x = \frac{P_{n+1} - P_n x_n}{Q_{n+1} - Q_n x_n} \tag{6}$$

with $P_{n+1}Q_n - Q_{n+1}P_n = 1$. It is easily shown by induction that

$$Q_n \geq P_n, \qquad Q_n - Q_{n-1} \geq P_n - P_{n-1} \tag{7}$$

for $n \geq 1$ and $\xi_n \equiv Q_n \xi \pmod{\mathbb{Z}}$. $Q_n$ is uniquely determined by this congruence. By using (6) and (7), we have

$$0 < \xi - \frac{P_n}{Q_n} = \frac{\xi_n/\xi_{n-1}}{Q_n(Q_n - (\xi_n/\xi_{n-1})Q_{n-1})} < \frac{1}{Q_n(Q_n - Q_{n-1})} \tag{8}$$

which gives an equality $\xi_n = Q_n\xi - P_n$ and also guarantees the convergence of (3). An interval exchange transform

$$\begin{cases} x \mapsto x + \xi & \text{if } x \in [0, 1-\xi) \\ x \mapsto x + \xi - 1 & \text{if } x \in [1-\xi, 1) \end{cases} \tag{9}$$

gives the rotation $x \mapsto x + \xi$ on the torus. The induced dynamics on $[0, \xi_1) \simeq \mathbb{R}/\xi_1\mathbb{Z}$ is given by the first return map

$$\begin{cases} x \mapsto x + \xi_2 & \text{if } x \in [0, \xi_1 - \xi_2) \\ x \mapsto x + \xi_2 - \xi_1 & \text{if } x \in [\xi_1 - \xi_2, \xi_1) \end{cases}.$$

In the similar manner, we have successive induced systems acting on $[0, \xi_n)$:

$$\Phi_{n+1} : \begin{cases} x \mapsto x + \xi_{n+1} & \text{if } x \in [0, \xi_n - \xi_{n+1}) \\ x \mapsto x + \xi_{n+1} - \xi_n & \text{if } x \in [\xi_n - \xi_{n+1}, \xi_n) \end{cases}$$

with $n = 0, 1, \ldots$ which give rotations of the smaller tori $[0, \xi_n) \simeq \mathbb{R}/\xi_n\mathbb{Z}$ of an angle $\xi_{n+1}$ with an initial value 0. The first return is described as:

$$\Phi_{n+1}(x) = \begin{cases} \Phi_n^{a_n}(x) & \text{if } x \in [0, \xi_n - \xi_{n+1}) \\ \Phi_n^{a_n-1}(x) & \text{if } x \in [\xi_n - \xi_{n+1}, \xi_n) \end{cases}$$

which gives a dynamical interpretation of the negative continued fraction. One may confirm the orbit of 0 by:

$$\Phi_{n+1}(0) = \Phi_n(\Phi_n^{a_n-1}(0)) = \Phi_n((a_n - 1)\xi_n) = (a_n - 1)\xi_n + \xi_n - \xi_{n-1} = \xi_{n+1}$$

and the structure of successive induced systems reveals a dynamical meaning of the number $Q_{n+1}$, that is, the smallest positive integer $M$ such that $M\xi \pmod{\mathbb{Z}}$ falls into $[0, \xi_n)$.

There is a simple way to convert the regular continued fraction

$$x = b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}.$$

into (3) and vice versa (c.f. Prop 1 in [27]). This is given by a rewriting rule of infinite words which transforms $b_1 b_2 \ldots$ into $a_1 a_2 \ldots$ from left to right:

$$\begin{cases} b_1 & \to b_1 + 1 \\ b_{2j} & \to 2^{b_{2j}-1} \\ b_{2j+1} & \to b_{2j+1} + 2 \end{cases}$$

where $j \geq 1$, $2^k = \overbrace{2\dots 2}^{k}$ and $2^0$ indicates the empty word $\lambda$. The converse rule is clearly:

$$\begin{cases} a_1 & \to a_1 - 1 \\ 2^k & \to k+1 \\ a_n \geq 3 & \to a_n - 2 \end{cases}$$

for $n \geq 2$. Especially, in view of the Lagrange theorem on regular continued fractions, the negative continued fraction expansion of $x$ is eventually periodic if and only if $x$ is real quadratic irrational.

The transformation $S$ has an infinite invariant measure $dx/(1-x)$ on $[0,1]$ and consequently Diophantine approximation by negative continued fraction is slow, which is also apparent from (8). However it matches better the induced system and the first return map than the regular continued fraction algorithm.

## 3. The case $\mu = 0$

We first prove Theorem 1 for $\mu = 0$. In this case the proof follows naturally by successive recoding the orbit $m\xi \in \mathbb{T}$ $(m = 0, 1, \dots)$ into the induced rotation $x \mapsto x + \xi_{n+1}$ acting on $[0, \xi_n)$. The clue of the proof is the fact that the number of necessary decomposition of intervals $[0, \xi_n)$ does not increase for $n \geq 1$.

Set $\mathcal{B}_0 = \mathcal{A}$, $J_0 = J$ and we recall that the rotation $\Phi_2 : x \mapsto x + \xi_2$ on $[0, \xi_1)$ is an induced system of the rotation $\Phi_1 : x \mapsto x + \xi_1$ on $[0, 1)$ by:

$$\Phi_2(x) = \begin{cases} \Phi_1^{a_1}(x) & \text{if } x \in [0, \xi_1 - \xi_2) \\ \Phi_1^{a_1 - 1}(x) & \text{if } x \in [\xi_1 - \xi_2, \xi_1) \end{cases}.$$

Therefore we define the recoding map $J_1 : [0, \xi_1) \to \mathcal{A}^*$ by

$$J_1(x) = \begin{cases} J_0(x)J_0(x+\xi_1)\dots J_0(x + (a_1 - 1)\xi_1) & \text{if } x \in [0, \xi_1 - \xi_2) \\ J_0(x)J_0(x+\xi_1)\dots J_0(x + (a_1 - 2)\xi_1) & \text{if } x \in [\xi_1 - \xi_2, \xi_1) \end{cases}$$

and put $\mathcal{B}_1 = J_1([0, \xi_1))$. We have $\mathcal{B}_1 \not\subset \mathcal{B}_0$, since $J_1(0)$ is of length $a_1 > 1$ as a word over $\mathcal{B}_0$. The function $J_1$ naturally extends to $J_1 : \mathbb{R}/\xi_1\mathbb{Z} \to \mathcal{B}_1$ by periodicity. As the map $J_0$ is discontinuous[5] at the set $\{w_i \mid i = 1, \dots, k\}$, the set of discontinuity of $J_1$ is given as a set $\{\omega_j^{(1)} \mid j = 0, 1, \dots, k_1 + 1\}$ with

$$0 = \omega_0^{(1)} < \omega_1^{(1)} < \omega_2^{(1)} < \dots < \omega_{k_1 - 1}^{(1)} < \omega_{k_1}^{(1)} < \omega_{k_1+1}^{(1)} = \xi_1$$

and each $\omega_j^{(1)}$ $(i = 1, \dots, k_1)$ has a form $\omega_u - N_u\xi_1$ $(u = 1, 2, \dots, k)$ where $N_u$ is the non negative integer that $\omega_u - N_u\xi_1 \in [0, \xi_1)$. From the definition of the first return map, this $N_u$ is a unique non negative integer less than $a_1$ that $\omega - N_u\xi_1$ (mod 1) falls into $[0, \xi_1)$, that is, $N_u = \lfloor \omega_u/\xi_1 \rfloor < a_1$. Note that there exists $j$ such that $\omega_j^{(1)} = \xi_1 - \xi_2$. Indeed $\omega_k - (a_1 - 1)\xi_1 = \xi_1 - (a_1\xi_1 - 1) = \xi_1 - \xi_2$. Thus we have $k_1 \leq k$ and the map $J_1$ has at most $k_1 + 1$ images, i.e., $^{\#}\mathcal{B}_1 \leq k_1 + 1$.

According to the usual convention, interval $[\omega_j^{(1)}, \omega_j^{(1)})$ is called $I_{J_1(y)}$ for $y \in [\omega_j^{(1)}, \omega_{j+1}^{(1)})$ since the name $J_1(y)$ does not depend on the choice of $y$. Then recalling

[5]This just means that its output word changes.

that $\Phi_2 : x \mapsto x + \xi_2$ on $[0, \xi_1)$ is an induced system of the rotation $\Phi_1 : x \mapsto x + \xi_1$ on $[0, 1)$. The original rotation word

$$J_0(0)J_0(\xi_1)J_0(2\xi_1)\dots$$

is decomposed into

$$J_0(0)J_0(\xi_1)\dots J_0((a_1-1)\xi_1)J_0(\xi_2)J_0(\xi_2+\xi_1)\dots = J_1(0)J_1(\xi_2)J_1(2\xi_2)\dots \in \mathcal{B}_1^{\mathbb{N}}.$$

We proceed in the similar manner. Assume $n \geq 2$ and we already defined $J_{n-1}$, $\mathcal{B}_{n-1}$ and the decomposition of $[0, \xi_{n-1})$ into

$$0 = \omega_0^{(n-1)} < \omega_1^{(n-1)} < \omega_2^{(n-1)} < \dots < \omega_{k_{n-1}-1}^{(n-1)} < \omega_{k_{n-1}}^{(n-1)} < \omega_{k_{n-1}+1}^{(n-1)} = \xi_{n-1}$$

with $k_{n-1} \leq k_{n-2}$. Further we may assume that there is a $j$ such that $\omega_j^{(n-1)} = \xi_{n-1} - \xi_n$. Since the rotation $x \mapsto x + \xi_{n+1}$ on $[0, \xi_n)$ is the induced system of the rotation $x \mapsto x + \xi_n$ on $[0, \xi_{n-1})$ through:

$$\Phi_{n+1}(x) = \begin{cases} \Phi_n^{a_n}(x) & \text{if } x \in [0, \xi_n - \xi_{n+1}) \\ \Phi_n^{a_n-1}(x) & \text{if } x \in [\xi_n - \xi_{n+1}, \xi_n), \end{cases}$$

we define the recoding map $J_n : [0, \xi_n) \to \mathcal{B}_{n-1}^*$ by

(10)

$$J_n(x) = \begin{cases} J_{n-1}(x)J_{n-1}(x+\xi_n)\dots J_{n-1}(x+(a_n-1)\xi_n) & \text{if } x \in [0, \xi_n - \xi_{n+1}) \\ J_{n-1}(x)J_{n-1}(x+\xi_n)\dots J_{n-1}(x+(a_n-2)\xi_n) & \text{if } x \in [\xi_n - \xi_{n+1}, \xi_n) \end{cases}$$

and put $\mathcal{B}_n = J_n([0, \xi_n))$. It follows $\mathcal{B}_n \not\subset \mathcal{B}_{n-1}$ from $a_n \geq 2$. Extend $J_n$ to a function from $\mathbb{R}/\xi_n\mathbb{Z}$ to $\mathcal{B}_n$ by periodicity. Now $J_{n-1}$ has $k_{n-1}+1$ points of discontinuity $\{\omega_j^{(n-1)} \mid j = 1, \dots, k_{n-1}+1\}$. The discontinuity of $J_n$ arises from these $k_{n-1}+1$ points together with $\xi_n - \xi_{n+1}$ and the end point $\xi_n$. The number of point of discontinuity of $J_n$ might increase to $k_{n-1}+3$ but it turns out that one can save 2 points. In fact such points are written down in a way:

$$0 = \omega_0^{(n)} < \omega_1^{(n)} < \omega_2^{(n)} < \dots < \omega_{k_n-1}^{(n)} < \omega_{k_n}^{(n)} < \omega_{k_n+1}^{(n)} = \xi_n$$

where each $\omega_j^{(n)}$ $(i = 1, \dots, k_n)$ has a form $\omega_u^{(n-1)} - N_u\xi_n$ $(u = 1, 2, \dots, k_{n-1}+1)$ and $N_u$ is the non negative integer that $\omega_u^{(n-1)} - N_u\xi_n \in [0, \xi_n)$, i.e., $N_u = \lfloor \omega_u^{(n-1)}/\xi_n \rfloor < a_n$. Firstly there is an index $s$ such that $\omega_s^{(n)} = \xi_n - \xi_{n+1}$ since $\xi_n - \xi_{n+1} = \xi_{n-1} - (a_n-1)\xi_n = \omega_j^{(n-1)} - (a_n-2)\xi_n$. Secondly $\omega_j^{(n-1)} = \xi_{n-1} - \xi_n$ and $\omega_{k_{n-1}+1}^{(n-1)} = \xi_{n-1}$ gives the same discontinuous point $\omega_s^{(n)}$ of $J_n$. These two coincidences show that $k_n \leq k_{n-1}$ and $^{\#}\mathcal{B}_n \leq k_n + 1$.

Set $I_{J_n(y)} = [\omega_j^{(n)}, \omega_{j+1}^{(n)})$ for $y \in [\omega_j^{(n)}, \omega_{j+1}^{(n)})$ since the name $J_n(y)$ does not depend on the choice of $y$. As $\Phi_{n+1} : x \mapsto x + \xi_{n+1}$ on $[0, \xi_n)$ is an induced system of $\Phi_n : x \mapsto x + \xi_n$ on $[0, \xi_{n-1})$. The word

$$J_{n-1}(0)J_{n-1}(\xi_n)J_{n-1}(2\xi_n)\dots$$

is decomposed into

$$\begin{aligned} &J_{n-1}(0)J_{n-1}(\xi_n)\dots J_{n-1}((a_n-1)\xi_n)J_{n-1}(\xi_{n+1})J_{n-1}(\xi_{n+1}+\xi_n)\dots \\ &= J_n(0)J_n(\xi_{n+1})J_n(2\xi_{n+1})\dots \in \mathcal{B}_n^{\mathbb{N}}. \end{aligned}$$

The case $\mu = 0$ is completed. $\square$

For the later use, we study in detail the map $J_n$ and the set $\mathcal{B}_n$. Let us introduce an order $<_{\text{lex}}$ in $\mathcal{A}^*$ for two elements $u = u_1 \dots u_s$ and $v = v_1 \dots v_t$. If $v$ is a proper prefix $u$ then $u <_{\text{lex}} v$. When $v$ is not a proper prefix of $u$ and $u$ is not a proper prefix of $v$, then take the first $i$ such that $u_i \neq v_i$ whenever $u \neq v$. In this case if $u_i < v_i$ (resp. $u_i > v_i$), then we say $u <_{\text{lex}} v$ (resp. $v <_{\text{lex}} u$). The order $<_{\text{lex}}$ is just a lexicographical order for words of the same length. The symbol $|u|$ stands for the length of the word $u$ and $u \leq_{\text{lex}} v$ means $u <_{\text{lex}} v$ or $u = v$. Then we can show a

**Proposition 1.** *The map $J_n : [0, \xi_n) \to \mathcal{B}_n$ preserves the order, i.e., $J_n(x) \leq_{\text{lex}} J_n(y)$ for $0 \leq x \leq y \leq \xi_n$ and consequently $J_n(\omega_i^{(n)}) <_{\text{lex}} J_n(\omega_{i+1}^{(n)})$. Especially ${}^{\#}\mathcal{B}_n = k_n + 1$. Exactly two different lengths $Q_{n+1}$ and $Q_{n+1} - Q_n$ appear in $\mathcal{B}_n$ for $n \geq 1$ and $Q_{n+1} = |J_n(0)|$ where $Q_n$ is defined by (5).*

Hereafter we classify words in $\mathcal{B}_n$ into *long words* and *short words* by their lengths.

*Proof.* By induction, we prove that $|J_n(x)|$ decreases only once at $x = \xi_n - \xi_{n+1}$. This is obvious for $n = 1$. By the definition (10), $|J_n(x)|$ decreases when either $x = \xi_n - \xi_{n+1}$ or $x + j\xi_n = \xi_{n-1} - \xi_n$ for $x \in [0, \xi_n)$ and $j = 0, \dots, a_n - 1$. One can confirm that the later case happens only when $j = a_n - 2$ and $x = \xi_n - \xi_{n+1}$. In other words, it turns out that discontinuities of $|J_n|$ and $|J_{n-1}|$ appear at the same point $x = \xi_n - \xi_{n+1}$. Thus, only two different lengths appear in $\mathcal{B}_n$. By the construction of the induced rotation, $|J_n(0)|$-times iteration of $\Phi_1$ gives the first return map $\Phi_n : [0, \xi_n) \to [0, \xi_n)$ through $x \mapsto x + \xi_{n+1}$. This implies $|J_n(0)| = Q_{n+1}$. In (10) for $x \in [0, \xi_n - \xi_{n+1})$, $x + j\xi_n \in [0, \xi_{n-1} - \xi_n)$ happens only when $j = a_n - 1$. Thus the length of the short word of $\mathcal{B}_{n-1}$ is $Q_{n+1} - (a_n - 1)Q_n = Q_n - Q_{n-1}$.[6] Now it is easy to show by induction that $J_n(x) \leq_{\text{lex}} J_n(y)$ for $0 \leq x \leq y \leq \xi_n$. $\square$

Before closing this section, we mention two extremal cases. As a general rotation word with an initial value 0 with respect to the $k$-block decomposition is recursively $(k+1)$-renewable, there seems no chance to get recursively 2-renewable aperiodic words. However having a closer look to the proof, if we start with the decomposition:

$$[0, 1) = [0, 1 - \xi) \cup [1 - \xi, 1)$$

then $1 = k - 1 = k_1 = k_2 = \dots$ and successive decompositions are

$$[0, \xi_n) = [0, \xi_n - \xi_{n+1}) \cup [\xi_n - \xi_{n+1}, \xi_n) \quad n = 1, 2, \dots$$

and it is recursively 2-renewable. This is nothing but the sturmian words.

Next we consider a general rotation word with an initial value 0 with respect to the $k$-block decomposition (1) with an additional condition:

$$\omega_i - \omega_j = A\xi + B \ (A, B \in \mathbb{Z}) \Longrightarrow i = j, A = B = 0.$$

(This condition is fulfilled when $\omega_i \in \mathbb{Q}$.) In this case, two $\omega_i^{(n-1)}, \omega_j^{(n-1)}$ $(i \neq j)$ produce different $\omega_u^{(n)}$ 's. In other words, $k_n$ does not decrease each step and $k = k_1 = k_2 = \dots$. In this case, we expect that the general rotation word is recursively $(k+1)$-renewable but not recursively $k$-renewable.

More detailed study on induced discontinuities is given in §9: with the help of Ostrowski's numeration system, we will be able to tell which discontinuities disappear by coincidence.

[6] One can also show this by induction.

## 4. The general case: $\mu \neq 0$

If $\mu = \omega_i$ for some $i$, the problem is transferred into a generalized rotation word of an angle $\xi$ and an initial value 0 with respect to the decomposition

$$\mathbb{T} = \bigcup_{i=0}^{k-1} [\omega_i - \mu, \omega_{i+1} - \mu). \tag{11}$$

Therefore the proof is exactly the same as that of the previous section.

For a general $\mu$, we first note that it is easy to show that the word is recursively $(k+2)$-renewable by the same technique. The problem is transferred into a generalized rotation word of an angle $\xi$ and an initial value 0 but the decomposition (11) needs to be subdivided at the origin and produces the decomposition of $\mathbb{T}$ into $k+1$ subintervals. Two generated subdivided intervals at the origin correspond to a same letter. However, assigning different two letters to these subdivided intervals, this general rotation word with respect to the $(k+1)$-block decomposition is recursively $(k+2)$-renewable by Theorem 1. Therefore the general rotation word is recursively $(k+2)$-renewable, since it is an image of the $(k+2)$-renewable word by a morphism which send the above two letters into one and the others to themselves (see Example 3). To show $(k+1)$-renewability, we should choose subintervals cleverly to have one less subdivisions.

Take a general rotation word $z$ of an angle $\xi = \xi_1$ and an initial value $\mu$ with respect to (1). Reviewing the proof for $\mu = 0$, if $\mu \in [0, \xi_1)$ then the orbit $\mu + i\xi_1$ ($i = 0, 1, \dots$) is recoded by the function $J_1$ and the word $z$ is $(k+1)$-renewable as a word over $\mathcal{B}_1$. Joining the above cases $\mu = \omega_i$, if $\mu \in \bigcup_{i=0}^{k-1} [\omega_i, \omega_i + \xi_1)$ then $z$ is $(k+1)$-renewable. However the set $\bigcup_{i=0}^{k-1} [\omega_i, \omega_i + \xi_1)$ may not exhaust $\mathbb{T}$. We construct a covering of $\mathbb{T}$ by such induced systems. It is sufficient to cover $[\omega_i, \omega_{i+1})$. Assume that $\omega_{i+1} - \omega_i > \xi_1$ since otherwise we have nothing to do. Put $q = \lfloor (\omega_{i+1} - \omega_i)/\xi_1 \rfloor$. Then for intervals $[\omega_i + (j-1)\xi_1, \omega_i + j\xi_1)$ for $j = 1, \dots, q$, the induced rotation is written as an interval exchange:

$$\begin{cases} x \mapsto x + \xi_2 & \text{if } x \in [\omega_i + (j-1)\xi_1, \omega_i + j\xi_1 - \xi_2) \\ x \mapsto x + \xi_2 - \xi_1 & \text{if } x \in [\omega_i + j\xi_1 - \xi_2, \omega_i + j\xi_1) \end{cases}.$$

For simplicity, take $i = 0$ and consider the interval $[0, \omega_1)$ with $q = \lfloor \omega_1/\xi \rfloor$. Then the function $J_1'$ is defined in the similar manner as $J_1$:

$$J_1'(x) = \begin{cases} J_0(x)J_0(x+\xi_1)\dots J_0(x+(a_1-1)\xi_1) & \text{if } x \in [(j-1)\xi_1, j\xi_1 - \xi_2) \\ J_0(x)J_0(x+\xi_1)\dots J_0(x+(a_1-2)\xi_1) & \text{if } x \in [j\xi_1 - \xi_2, j\xi_1) \end{cases}.$$

The set of discontinuity of $J_1'$ is just a shifted set of those of $J_1$:

$$\{\omega_u^{(1)} + (j-1)\xi_1 \mid u = 1, \dots, k_1 + 1\}$$

and we have

$$\overbrace{00\dots0}^{j-1} J_1'(x) = J_1(x - (j-1)\xi_1) \overbrace{00\dots0}^{j-1}. \tag{12}$$

This means that if $x \in [(j-1)\xi_1, j\xi_1)$ then $z$ is $(k_1+1)$-renewable ($k_1 \le k$) over the words $\overbrace{0^{-1}0^{-1}\dots0^{-1}}^{j-1} \mathcal{B}_1 \overbrace{00\dots0}^{j-1}$ where the symbol $0^{-1}$ indicates the removal of

0 from the prefix. Now we have shown that if $\mu$ lies in

$$\bigcup_{i=0}^{k-1} \bigcup_{j=1}^{\lfloor(\omega_{i+1}-\omega_i)/\xi\rfloor} [\omega_i + (j-1)\xi_1, \omega_i + j\xi_1),$$

then $z$ is $(k_1+1)$-renewable. The remainder set to be covered is

$$\bigcup_{i=0}^{k-1} \left[\omega_i + \left\lfloor \frac{\omega_{i+1}-\omega_i}{\xi_1} \right\rfloor \xi_1,\ \omega_{i+1}\right).$$

This job is completed by considering the induced system on $[\omega_{i+1}-\xi_1, \omega_{i+1})$ for $i = 0, \dots, k-1$. The construction is done in the similar manner. The induced rotation is given as an interval exchange:

$$\begin{cases} x \mapsto x + \xi_2 & \text{if } x \in [\omega_{i+1}-\xi_1, \omega_{i+1}-\xi_2) \\ x \mapsto x + \xi_2 - \xi_1 & \text{if } x \in [\omega_{i+1}-\xi_2, \omega_{i+1}) \end{cases}.$$

Take $i = k$ for simplicity. We may assume $\omega_{k-1} < 1 - \xi_1$ since otherwise the covering is already over. One can similarly define

$$J_1''(x) = \begin{cases} J_0(x)J_0(x+\xi_1)\dots J_0(x+(a_1-1)\xi_1) & \text{if } x \in [1-\xi_1, 1-\xi_2) \\ J_0(x)J_0(x+\xi_1)\dots J_0(x+(a_1-2)\xi_1) & \text{if } x \in [1-\xi_2, 1) \end{cases}$$

and the set of discontinuity of $J_1''$ is just a shifted set of those of $J_1$:

$$\{\omega_u^{(1)} - \xi_1 \in \mathbb{T} \mid u = 1, \dots, k_1+1\}.$$

Similarly as (12), for each $x \in [1-\xi_1, 1)$, we have $b \in \mathcal{A}$ such that

$$J_1''(x)b = (k-1)J_1(x+\xi_1)$$

and $b$ is determined by $x + \xi_1 \in [\omega_b, \omega_{b+1})$. Thus in the same way, when $\mu \in [\omega_{i+1}-\xi_1, \omega_{i+1})$, $z$ is $(k_1+1)$-renewable and we have shown that for any $\mu \in T$ the general rotation word $z$ is $(k+1)$-renewable. Note that this proof shows that there are two choices of the set $\mathcal{B}_1 \subset \mathcal{A}^*$ for

$$\mu \in \left[\omega_{i+1} - \omega_i, \omega_i + \left\lfloor \frac{\omega_{i+1}-\omega_i}{\xi_1} \right\rfloor \xi_1\right).$$

Indeed, usually we have several choices of induced systems. The only requirement is that the interval should be decomposed into at most $k+1$ parts.

Now we have shown that for any $\mu$ the general rotation word $z$ is $(k+1)$-renewable. However this procedure can be repeated recursively to the subinterval (strictly speaking, the subsystem) to which $\mu$ belongs. As we have seen that the set of discontinuity of $J_1'$ and $J_1''$ are given by translations of those of $J_1$, by the same reason as the case $\mu = 0$, the number of decomposition of subintervals does not exceed $k+1$. Therefore $z$ is recursively $(k+1)$-renewable and the proof is finished. □

The recursive covering of $[0, 1)$ constructed in this §4 will be reused in §10.

## 5. An Example

Let us show a sample computation of a general rotation word of an angle $\xi = 2^{-2/3}$ and an initial value $\mu = 0$ with respect to a decomposition $[0, 1) = [0, 1/3) \cup [1/3, 1)$

$$0101101101011011011110110110101101101011011011110110110\ldots$$

following the proof of Theorem 1.

$$\begin{aligned}
[0, \xi_1) &= [0, 1/3) \ \cup \ [1/3, 1 - 2^{-2/3}) \ \cup \ [1 - 2^{-2/3}, 2^{-2/3}) = I_{01} \ \cup \ I_{11} \cup I_1 \\
[0, \xi_2) &= [0, 4/3 - 2^{1/3}) \ \cup \ [4/3 - 2^{1/3}, 2 - 3 \cdot 2^{-2/3}) \ \cup \ [2 - 3 \cdot 2^{-2/3}, -1 + 2^{1/3}) \\
&= I_{01011} \ \cup \ I_{01111} \ \cup \ I_{011} \\
[0, \xi_3) &= [0, 4/3 - 2^{1/3}) \ \cup \ [4/3 - 2^{1/3}, 2 - 3 \cdot 2^{-2/3}) \ \cup \ [2 - 3 \cdot 2^{-2/3}, -3 + 5 \cdot 2^{-2/3}) \\
&= I_{01011011} \cup I_{01111011} \cup I_{011} \\
[0, \xi_4) &= [0, 12 - 19 \cdot 2^{-2/3}) \ \cup \ [12 - 19 \cdot 2^{-2/3}, 19/3 - 5 \cdot 2^{1/3}) \cup \\
& \quad [19/3 - 5 \cdot 2^{1/3}, -5 + 4 \cdot 2^{1/3}) \\
&= I_{010110110101101101111011011} \cup I_{0101101101011011011} \cup I_{0101101101111011011}
\end{aligned}$$

In fact, this method gives a rapid algorithm for computation of general rotation words. The recursive construction algorithm for $\mu \neq 0$ in §4 may not be suitable in reality because we have to switch to other systems, keeping in memory all systems appear in the process. It is easier to shift the origin at the expense of one more subdivision, as explained in the beginning of §4.

## 6. Quadratic rotations are primitive substitutive

In this section, one direction of Theorem 2 is treated. We prove that a general rotation word of a quadratic irrational angle $\xi$ and an initial value $\mu$ with respect to the decomposition (1) is primitive substitutive provided $\omega_i - \mu \in \mathbb{Q}(\xi)$ for all $i$. By the same technique stated in the beginning of §4, it is sufficient to prove this fact for $\mu = 0$ and $\omega_i \in \mathbb{Q}(\xi)$ since it just amounts to increasing by 1 the number of subdivisions. Moreover, we may assume that $\omega_i = 1 - \xi$ for some $i$ by the same reason. Under these assumptions, the proof of Theorem 1 reads that the irrational rotation $x \mapsto x + \xi_1$ on $\mathbb{T}$ gives rise to induced rotations $x \mapsto x + \xi_{n+1}$ on $[0, \xi_n)$ for $n = 1, 2, \ldots$ and the set of discontinuity $\{\omega_i \mid i = 1, 2, \ldots, k\}$ is transformed into $\{\omega_i^{(n)} \mid i = 1, 2, \ldots, k\}$[7] with $\omega_i^{(n)} = \xi_n - \xi_{n+1}$ for some $i$. Each $\omega_i^{(n)}$ has a form $\omega_u^{(n-1)} - \lfloor \omega_u^{(n-1)} / \xi_n \rfloor \xi_n$ $(u = 1, 2, \ldots, k)$. Let us renormalize the sizes of the rotations and observe the orbit of $\omega_i$. The $n$-th induced system $[0, \xi_n) \ni x \mapsto x + \xi_{n+1} \in [0, \xi_n)$ is renormalized into:

$$[0, 1) \ni x \mapsto x + x_{n+1} \in [0, 1)$$

with $x_{n+1} = \xi_{n+1}/\xi_n$ and the discontinuous points must be $\rho_i^{(n)} = \omega_i^{(n)}/\xi_n$. Therefore $\rho_i^{(n)}$ should have a form $\rho_u^{(n-1)}/x_n - \lfloor \rho_u^{(n-1)}/x_n \rfloor$ for some $u$. Moreover, we have $x_n = S^n(\xi)$ $(n = 0, 1, \ldots)$ by the negative continued fraction map $S$ as explained in §2. For $\beta > 1$, we define the $\beta$-transform by

$$T_\beta : [0, 1) \ni x \mapsto \beta x - \lfloor \beta x \rfloor \in [0, 1).$$

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[7]The cardinality of $\{\omega_i^{(n)} \mid i = 1, 2, \ldots, k\}$ could be less than $k$ by coincidences $\omega_i^{(n)} = \omega_{i+1}^{(n)}$.

Then the set of discontinuity of $n$-th renormalized induced rotation is written as

$$\left\{ T_{1/x_{n-1}} \dots T_{1/x_0}(\omega_i) \;\middle|\; i = 1, 2, \dots, k \right\}.$$

Let us denote by $\upsilon_n(y) = T_{1/x_{n-1}} \dots T_{1/x_0}(y)$ for $n = 1, 2, \dots$ with $x_n = S^n(\xi)$. The clue of the proof is to show a

**Proposition 2.** *If $\xi \in (0,1)$ is quadratic irrational and $y \in \mathbb{Q}(\xi)$ then the sequence $(\upsilon_n(y))_{n=1,2,\dots}$ is eventually periodic.*

In view of Proposition 4 given in §9, this is a generalization of the result in [21] on Ostrowski expansions. Once Proposition 2 is established, it is easy to see that the quadratic rotation word is primitive substitutive. Indeed, taking the least common multiple $L$ of periods of $k$ sequences $(\upsilon_n(\omega_i))_{n=0,1,\dots}$, there exists an integer $m \geq 0$ such that the $m$-th renormalized induced system and $(m+L-1)$-th renormalized induced system is exactly the same. Hence the associated rotation word $z^{(m)} \in \mathcal{B}_m^{\mathbb{N}}$ is successively decomposed $(L-1)$-times into $k$-blocks and gives rise to $z^{(m+L-1)} \in \mathcal{B}_{m+L}^{\mathbb{N}}$ which has the same structure as $z^{(m)}$. Therefore $z^{(m)}$ is a fixed point of the substitution on $\mathcal{B}_m$. This substitution is primitive, because repeated application of the substitution to $\mathcal{B}_m$ yields sufficiently long words generated by the induced rotation $x \mapsto x + \xi_{m+1}$, each of which must contain all letters of $\mathcal{B}_m$, by the minimality of the irrational rotation. This shows that $z^{(1)}$ is primitive substitutive.

We give two proofs of Proposition 2. A technical difficulty arises from the fact that each $1/x_i$ is not necessary an algebraic integer. In the first proof, we choose periodically varying $\mathbb{Z}$-bases to overcome it.

6.1. **The first proof.** According to the last remark of §2, $(x_n)_{n=0,1,\dots}$ is eventually periodic. Without loss of generality, we may assume that the negative continued fraction of $\xi$ is purely periodic, i.e., there is a positive integer $L$ that $x_n = x_{n+L}$ for $n \geq 0$. By (6), there is $(P_L, P_{L+1}, Q_L, Q_{L+1}) \in \mathbb{N}^4$ depending on $x_n$ such that

$$x_n = (P_{L+1} - P_L x_n)/(Q_{L+1} - Q_L x_n). \tag{13}$$

Put $h(y) = Q_L y^2 - (Q_{L+1} + P_L)y + P_{L+1}$. By (7), we see $h(x_n) = 0, h(0) = P_{L+1} > 0, h(1) = (P_{L+1} - P_L) - (Q_{L+1} - Q_L) \leq 0$ and the conjugate $x_n'$ can not be less than 1. $x_n' \notin \mathbb{Q}$ implies $x_n' > 1$. Let $\eta = \min_n x_n' > 1$. The general term of the sequence is

$$\upsilon_n(y) = \frac{y - \sum_{k=0}^{n-1} \lfloor \upsilon_k(y)/x_k \rfloor x_0 x_1 \dots x_k}{x_0 x_1 \dots x_{n-1}}. \tag{14}$$

Note that by periodicity of $(x_n)_{n=0,1,\dots}$, $\lfloor \upsilon_k(y)/x_k \rfloor \leq \lfloor 1/x_k \rfloor$ are uniformly bounded by a positive constant $L$. Therefore

$$|(\upsilon_n(y))'| \leq \frac{|y'|}{\eta^n} + \sum_{k=0}^{n-1} \frac{L}{\eta^{n-k-1}} \leq |y'| + \frac{L}{1-\eta}.$$

This shows that $(\upsilon_n(y), (\upsilon_n(y))') \in \mathbb{R}^2$ is bounded. On the other hand there exist a positive integer $M$ and $(p_n, q_n) \in \mathbb{Z}^2$ such that $\upsilon_n(y) = (p_n + q_n x_n)/M$. Indeed, the case $n = 0$ is trivial and

$$\begin{aligned} \upsilon_{n+1}(y) &= \frac{\upsilon_n(y)}{x_n} - c_n = \frac{1}{M}\left(\frac{p_n}{x_n} + q_n\right) - c_n \\ &= \frac{1}{M}(p_n(a_n - x_{n+1}) + q_n) - c_n = \frac{p_{n+1} + q_{n+1}x_{n+1}}{M} \end{aligned}$$

with $c_n = \lfloor \upsilon_n(y)/x_n \rfloor$. Again by periodicity of $(x_n)_{n=0,1,\dots}$ and the boundedness of $(\upsilon_n(y), (\upsilon_n(y))')$, we see that $(p_n, q_n) \in \mathbb{Z}^2$ is bounded and this implies that the sequence $(\upsilon_n(y))_{n=1,2,\dots}$ is eventually periodic. $\square$

The second proof is based on [8] and [34] where they treated beta expansions in a Pisot number base.

6.2. **The second proof.** We may assume that there is a positive integer $L$ that $x_n = x_{n+L}$ for $n \geq 0$. By (6),

$$x = \frac{P_{L+1} - xP_L}{Q_{L+1} - xQ_L}$$

with $P_{L+1}Q_L - Q_{L+1}P_L = 1$. Putting $\kappa = Q_{L+1} - xQ_L$, we have

$$\kappa^2 - (Q_{L+1} - P_L)\kappa + 1 = 0.$$

From (5) and $a_n = x_n + 1/x_{n-1}$, we deduce

$$Q_{n+1} - x_nQ_n = \frac{Q_n - x_{n-1}Q_{n-1}}{x_{n-1}} = \dots = \frac{Q_1 - x_0Q_0}{x_0x_1\dots x_{n-1}} = \frac{1}{x_0x_1\dots x_{n-1}}.$$

Therefore

$$\kappa = Q_{L+1} - xQ_L = \frac{1}{x_0x_1\dots x_{L-1}} \tag{15}$$

is a quadratic unit with $\kappa > 1$ and $|\kappa'| < 1$, i.e., a quadratic Pisot unit. Take a finite set $D = \{\sum_{k=0}^{L-1} c_kx_0\dots x_k \mid 1/x_k > c_k \in \mathbb{Z}\}$ and choose a positive integer $U$ such that all $dU$ are algebraic integers for $d \in D$. Then in view of (14), we can write

$$\upsilon_{ML-1}(y) = \kappa^My - \sum_{i=0}^{M-1} d_i\kappa^i$$

with $d_i \in D$. Then $(U\upsilon_{ML-1}(y))_{M=1,2\dots}$ is a sequence of algebraic integers and one can easily show that $(U\upsilon_{ML-1}(y), (U\upsilon_{ML-1}(y))')$ is bounded in $\mathbb{R}^2$ in a similar manner as in the first proof. Therefore the sequence $(\upsilon_{ML-1}(y))_{M=1,2,\dots}$ is eventually periodic and this proves that $(\upsilon_n(y))_{n=1,2,\dots}$ is eventually periodic. $\square$

In his undergraduate text book ([35], page 212), Takagi emphasized that the denominator like $\kappa$ of the modular equivalence equation (13) becomes a unit. The product relation (15) seems interesting in its own right.

## 7. Recovering parameters of rotation words

From a given general rotation word $z$ on $\{0, 1, \dots, k-1\}$ with $k \geq 2$, there is a way to recover all parameters, i.e, the angle $\xi$, the initial value $\mu$, and the decomposition of the unit interval $[0, 1)$. For the later use, we briefly describe this method in this section. A lot of methods had been discussed for sturmian words in relation to continued fraction expansions. One can find a nice survey in Chapter 6 of [18] in which a dynamical and algorithmic way of this recoding is discussed in detail. Here we only mention a 'transcendental' way to recover such parameters, which might be a folklore.

First of all, from unique ergodicity of the irrational rotation, the decomposition of $[0, 1)$ is recovered immediately since it amounts to computing frequencies of letters of $\{0, 1, \dots, k-1\}$. Thus we have given (1) with $0 = \omega_0 < \omega_1 < \dots < \omega_{k-1} < \omega_k = 1$ and we may assume that the angle $\xi$ is in $[-1/2, 1/2)$. Choose an arbitrary $j \in \{1, 2, \dots, k-1\}$ and recode the word $z = z_0z_1\dots$ into $z' = z'_0z'_1\dots$ where $z'_i = 0$

for $z_i < j$ and $z'_i = 1$ for $z_j \geq j$. Assume first that $\omega_j > 1/2$. Then we observe in this rotation word $z'$ at most two different longest runs of 0's, say, $0^M$ and $0^{M+1}$. This implies that $\omega_j/(M+1) \leq \xi \leq \omega_j/M$ with $M \geq 1$. We get frequencies of $0^M$ and $0^{M+1}$ which are denoted by $\kappa_M$ and $\kappa_{M+1}$ (one of them could be zero). When $\xi \in [0, 1/2)$, consider a decomposition

$$[0, \xi) = [0, \omega_j - M\xi) \cup [\omega_j - M\xi, \xi) = I_{M+1} \cup I_M.$$

Then $\mu + n\xi \pmod 1 \in I_{M+1}$ is the beginning of the $M+1$ runs of zeroes and $\mu + n\xi \pmod 1 \in I_M$ is the beginning of the $M$ runs of zeroes. Therefore, we have

$$\frac{\kappa_{M+1}}{\kappa_M} = \frac{\omega_j - M\xi}{(M+1)\xi - \omega_j}.$$

From this equality, we got to know the angle:

$$\xi = \frac{\omega_j(\kappa_M + \kappa_{M+1})}{(M+1)\kappa_{M+1} + M\kappa_M}.$$

The case $\xi \in [-1/2, 0)$ is likewise and the final formula is the same. Secondly if $\omega_j < 1/2$, then we observe in $z'$ at most two different runs of 1's. The later discussion is the similar and we obtain:

$$\xi = \frac{(1-\omega_j)(\kappa_M + \kappa_{M+1})}{(M+1)\kappa_{M+1} + M\kappa_M}.$$

Our final task is to get the initial value. For $z = z_0 z_1 \dots$, take an increasing integer sequence of occurrences of the letter 0, i.e., $0 = n_0 < n_1 < \dots$ with $z_{n_j} = 0$. Then we have

$$\mu + n_j\xi \pmod 1 \in [0, \omega_1)$$

and hence

$$\mu \in \bigcap_{j=0}^{\infty} ([-n_j\xi, \omega_1 - n_j\xi) \pmod 1)$$

The intersection should be a single point, since it is non empty and $n\xi \pmod 1$ for $n = 1, 2, \dots$ are uniformly distributed in $\mathbb{T}$. Therefore we finally recovered the parameter $\mu$. $\square$

It might be a problem that without information of the natural order of letters, this recovered parameters are unique or not.

## 8. Uniqueness of decomposition

As we stated in the last part of §4, the choice of $k+1$ blocks of general rotation words is not unique. On the other hand, in this section we show that once we have chosen $k+1$ blocks as in the proof of Theorem 1, the way to decompose the general rotation word into these blocks $\mathcal{B}_n$ is unique when $n$ is large. Such $\mathcal{B}_n$ is called a *code*. This uniqueness will be used in the proof of the remaining direction of Theorem 2.

**Proposition 3.** *Let $J_n(\omega_u^{(n)}) = s_1 \dots s_{Q_{n+1}} \in \mathcal{B}_n$ be a long word in the sense of Proposition 1. Assume that there are $i, j$ such that $s_i \neq s_j$. The we have*

$$[\omega_u^{(n)}, \omega_{u+1}^{(n)}) = \bigcap_{i=1}^{Q_{n+1}} (J^{-1}(s_i) - (i-1)\xi) = \bigcap_{i=1}^{Q_{n+1}} [\omega_{s_i} - (i-1)\xi, \omega_{s_i+1} - (i-1)\xi) \quad \textit{in } \mathbb{T}.$$

Clearly, if $n$ is sufficiently large, the long word will contain two letters and satisfy the assumption of Proposition 3.

*Proof.* Note that $y \in \bigcap_{i=1}^{\ell}(J^{-1}(s_i) - (i-1)\xi)$ is equivalent to $J(y + (i-1)\xi) = s_i$ for $i = 1, \dots, \ell$. Hence by construction of $\omega_i^{(n)}$, the inclusion

$$[\omega_u^{(n)}, \omega_{u+1}^{(n)}) \subset \bigcap_{i=1}^{Q_{n+1}} (J^{-1}(s_i) - (i-1)\xi)$$

is obvious. By the assumption, $(J^{-1}(s_i) - (i-1)\xi) \cap (J^{-1}(s_j) - (j-1)\xi)$ is an interval, since the sum of length of two intervals does not exceed 1. Thus we can put

$$\bigcap_{i=1}^{Q_{n+1}} (J^{-1}(s_i) - (i-1)\xi) = [t_1, t_2), \quad t_1 \le \omega_u^{(n)} \le \omega_{u+1}^{(n)} \le t_2.$$

Let us show $t_2 = \omega_{u+1}^{(n)}$. Recall that $\omega_{u+1}^{(n)}$ is a discontinuous point of the $n$-th induced rotation on $[0, \xi_n)$ and $J_n(\omega_u^{(n)}) <_{\text{lex}} J_n(\omega_{u+1}^{(n)})$. If $|J_n(\omega_u^{(n)})| = |J_n(\omega_{u+1}^{(n)})|$ then $t_2 \le \omega_1^{(n)}$ since $J_n(\omega_u^{(n)})$ and $J_n(\omega_{u+1}^{(n)}) = s'_1 \dots s'_{Q_{n+1}}$ are different words of the same length and hence

$$\left( \bigcap_{i=1}^{Q_{n+1}} (J^{-1}(s_i) - (i-1)\xi) \right) \cap \left( \bigcap_{i=1}^{Q_{n+1}} (J^{-1}(s'_i) - (i-1)\xi) \right) = \emptyset$$

which implies $t_2 \le \omega_{u+1}^{(n)}$. Next assume that $|J_n(\omega_u^{(n)})| > |J_n(\omega_{u+1}^{(n)})|$. Reviewing the proof of Theorem 1, this happens only when $\omega_{u+1}^{(n)} = \xi_n - \xi_{n+1}$ and $p(J_n(\omega_u^{(n)})) \le_{\text{lex}} J_n(\omega_{u+1}^{(n)})$. In this case, consider words

$$J(x)J(x+\xi)\dots J(x + (Q_{n+1} - 1)\xi)$$

for $x \in [\omega_u^{(n)}, \omega_{u+1}^{(n)}]$. If $x \in [\omega_u^{(n)}, \omega_{u+1}^{(n)})$, the words are the same and

$$J(\omega_{u+1}^{(n)} + (Q_{n+1} - Q_n)\xi) = J(\xi_n - \xi_{n+1} - (\xi_{n+1} - \xi_n)) = J(0) = 0$$

and therefore

$$k - 1 = J(\omega_{u+1}^{(n)} + (Q_{n+1} - Q_n)\xi - \varepsilon) \ne J(\omega_{u+1}^{(n)} + (Q_{n+1} - Q_n)\xi) = 0$$

for a sufficiently small positive $\varepsilon$.[8] This shows that

$$J(\omega_u^{(n)})J(\omega_u^{(n)}+\xi)\dots J(\omega_u^{(n)}+(Q_{n+1}-1)\xi) \ne J(\omega_{u+1}^{(n)})J(\omega_{u+1}^{(n)}+\xi)\dots J(\omega_{u+1}^{(n)}+(Q_{n+1}-1)\xi)$$

and $t_2 \le \omega_{u+1}^{(n)}$.

Finally we prove that $t_1 = \omega_u^{(n)}$. Note that $\omega_0^{(n)} = 0$ for all $n$. Therefore if $u = 0$, then by using $J^{-1}(0) = [0, \omega_1)$, we have $t_1 \ge 0 = \omega_0^{(n)}$ and the proof is completed for $u = 0$. When $u > 0$ by the proof of Proposition 1, the predecessor of a long word in $\mathcal{B}_n$ must be long and we have $|J_n(\omega_{u-1}^{(n)})| = |J_n(\omega_u^{(n)})|$. Thus we have $J_n(\omega_{u-1}^{(n)}) \ne J_n(\omega_u^{(n)})$ and $t_1 \ge \omega_u^{(n)}$ in a similar manner. □

[8] Considering the order $<_{\text{lex}}$ on these words, there should be some $j < Q_{n+1} - Q_n$ such that $J(\omega_u^{(n)} + j\xi) < J(\omega_{u+1}^{(n)} + j\xi)$ indeed.

Let us extend general rotation words to bi-infinite words. Fix $\xi \in [0,1) \setminus \mathbb{Q}$, $\mu \in [0,1)$ and the decomposition (1). Then a bi-infinite general rotation word is naturally defined by

$$z = \dots J(\mu - 2\xi)J(\mu - \xi)J(\mu)J(\mu + \xi)J(\mu + 2\xi) \dots \in \{0, 1, \dots, k-1\}^{\mathbb{Z}}.$$

Then $z$ is $(k+1)$-renewable by blocks $\mathcal{B}_n = J_n([0, \xi_n))$ in the similar manner.

**Theorem 3.** *Assume that each long word of $\mathcal{B}_n$ has at least two letters of $\mathcal{A}$, $k \leq Q_{n+1} - Q_n$ and $\xi_n \leq \omega_v \leq 1 - \xi_n + \xi_{n+1}$ with some $v \in \{1, \dots, k-1\}$. Then there is only one way to decompose a generalized rotation word $z$ into $\mathcal{B}_n$.*

From this Proposition, the decomposition of the original rotation word is unique, because for a given one-sided general rotation word, the extension to a bi-infinite rotation word is unique (c.f. §7).

The length of short words of $\mathcal{B}_n$ is not less than $\prod_{i=1}^{n}(a_i - 1)$. Hence assumptions of Theorem 3 are fulfilled[9] for a sufficiently large $n$. The condition $k \leq Q_{n+1} - Q_n$ assures that the length of the short word is not less than the cardinality of short words. This condition is necessary. For instance, $\mathcal{B}_1 = \{01, 0, 1\}$ holds for the decomposition $[0, 1/2) \cup [1/2, 1)$ with $\xi > 1/2$.

*Proof.* We say that two words $z_i z_{i+1} \dots, z_{i+\ell}$ and $z_j z_{j+1} \dots, z_{j+\ell'}$ overlapped if there are $i, j$ such that $i < j < i + \ell$. Write

$$z = \dots z_{-2} z_{-1} z_0 z_1 z_2 \dots$$

with $z_i = J(\mu + i\xi) \in \mathcal{A}$. First we give a standard algorithm to decompose $z$ into $\mathcal{B}_n$. Put $\ell = Q_{n+1} - 1$ for simplicity. Consider a set

$$L = \{i \in \mathbb{Z} \mid z_i z_{i+1} \dots, z_{i+\ell} \text{ is a long word of } \mathcal{B}_n\} \subset \mathbb{Z}.$$

Since $z_i z_{i+1} \dots, z_{i+\ell} = s$ is a long word, by Proposition 3 we have $\mu + i\xi \pmod 1 \in [\omega_u^{(n)}, \omega_{u+1}^{(n)})$ with $s = J_n(\omega_u^{(n)})$. This implies that $\mu + i\xi \pmod 1 \in [\omega_u^{(n)}, \omega_{u+1}^{(n)}) \subset [0, \xi_n)$ and the first return map to the interval $[0, \xi_n)$ is realized by $x \to x + (\ell+1)\xi$. This means that $\mu + k\xi \pmod 1 \notin [0, \xi_n)$ for $k = 1, 2, \dots, \ell$ and hence $j \neq k$ for $k = 1, 2, \dots, \ell$. Therefore there are no overlap of long words in $z$. In this manner we can first decide all locations of long words in $z$. Let $z_i \dots z_{i+\ell}$ and $z_j \dots z_{j+\ell}$ be above decided two long words with $i + \ell < j$ and assume that there are no $k$ such that $i + \ell < k < j$ and $k \in L$. If $i + \ell + 1 \neq j$, consider the word $z_{i+\ell+1} \dots z_{j-1}$. $\mu + (i + \ell + 1)\xi \in [0, \xi_n)$ implies that $J_n(\mu + (i + \ell + 1)\xi)$ is a prefix of $z_{i+\ell+1} \dots$ and we must have $i + \ell + |J_n(\mu + (i + \ell + 1)\xi)| \leq j - 1$ since otherwise this short word $J_n(\mu + (i + \ell + 1)\xi)$ and the long word $z_j \dots z_{j+\ell}$ overlap and we would have $\mu + (i + \ell + |J_n(\mu + (i + \ell + 1)\xi)| + 1)\xi \pmod 1 \in [0, \xi_n)$ , contradicting the property of the first return map again. Iterating this, $z_{i+\ell+1} \dots z_{j-1}$ is decomposed into short words. Therefore we have a decomposition of $z$ into $\mathcal{B}_n$: $z = \dots x_{-2} x_{-1} x_0 x_1 x_2 \dots$ with $x_k \in \mathcal{B}_n$. Let us say that this is the *standard decomposition.*

We claim that any decomposition $z = \dots y_{-2} y_{-1} y_0 y_1 y_2 \dots$ with $y_i \in \mathcal{B}_n$ and any $K > 0$, there exist $i \leq -K$ and $j \geq K$ such that $y_i$ and $y_j$ are long words of $\mathcal{B}_n$. In fact, for e.g., assume that $y_i$ is a short word for all $i \leq -K$ and $y_{-K} = z_j \dots z_{j+\ell}$. This would mean that any factor of length $\ell + 1$ appears in the left infinite word $\dots z_{j+\ell-1} z_{j+\ell}$ must be a short word, since

$$\{\mu + (j - (\ell + 1)t)\xi \pmod 1 \mid t = 0, 1, \dots\}$$

[9] Recall that there are infinitely many $i$'s with $a_i > 2$.

is dense in $[0,1)$. However, because the cardinality of short words is not greater than $\ell+1$, the word $\dots z_{j+\ell-1}z_{j+\ell}$ must be periodic (for e.g., see Proposition 1.1.1. in [18]). Then $\xi$ would be rational, a contradiction. This proves the claim. Define

$$C(\dots x_{-2}x_{-1}x_0x_1x_2\dots) = \{i \in \mathbb{Z} \mid \exists t\ x_t = z_i \dots z_j \text{ is a long word}\}$$

and

$$C(\dots y_{-2}y_{-1}y_0y_1y_2\dots) = \{i \in \mathbb{Z} \mid \exists t\ y_t = z_i \dots z_j \text{ is a long word}\}.$$

By the above claim, $C(\dots y_{-2}y_{-1}y_0y_1y_2\dots)$ is neither bounded from below nor from above. From the definition of the standard decomposition,

$$C(\dots y_{-2}y_{-1}y_0y_1y_2\dots) \subset C(\dots x_{-2}x_{-1}x_0x_1x_2\dots).$$

It is sufficient to show that these two sets are equal. Indeed, if $y_i = z_{i_1}\dots z_{i_2}$ and $y_j = z_{j_1}\dots z_{j_2}$ are adjacent long words with $i<j$, the decomposition of word $z_{i_2+1}\dots z_{j_1-1}$ into short words is trivially unique.

Assume that there exists

$$k \in C(\dots x_{-2}x_{-1}x_0x_1x_2\dots) \setminus C(\dots y_{-2}y_{-1}y_0y_1y_2\dots).$$

Choose $i \in C(\dots y_{-2}y_{-1}y_0y_1y_2\dots)$ with $i<k$ and find a minimum

$$k_1 \in C(\dots x_{-2}x_{-1}x_0x_1x_2\dots) \setminus C(\dots y_{-2}y_{-1}y_0y_1y_2\dots)$$

with $i<k_1$. Then the decomposition of $z_{i+\ell+1}\dots z_{k_1-1} \in \mathcal{A}^*$ into $\mathcal{B}_n$ is unique, since there are no occurrence of long words any more. This shows that the long word $z_{k_1}\dots z_{k_1+\ell}$ has a short word prefix $z_{k_1}\dots z_{k_1+\ell'}$. In view of Proposition 1, this happens only when $z_{k_1}\dots z_{k_1+\ell}$ is a long word $J_n(x)$ with $x \in [0,\xi_n-\xi_{n+1})$ and $z_{k_1}\dots z_{k_1+\ell'}$ is the smallest short words with respect to the order $<_{\text{lex}}$. Thus $z_{k_1+\ell'+1} = J(x+\ell'\xi)$ is the prefix of a short word. Since $\ell' = Q_{n+1}-Q_n$, we have

$$J(x+\ell'\xi) = J(x+(Q_{n+1}-Q_n)\xi) = J(x+\xi_{n+1}-\xi_n) \geq v$$

because $x+\xi_{n+1}-\xi_n \in [-\xi_n+\xi_{n+1},0)$. However by the assumption, each short word of $\mathcal{B}_n$ begins with a letter less than $v$, which gives a contradiction. □

## 9. Ostrowski's numeration system and induced discontinuities

Ostrowski's numeration system has deep connection to the distribution of $(n\xi)_{n=1,2,\dots}$ in $\mathbb{T}$ and combinatorics on words: a survey is found in [4]. It is extensively used in recoding sturmian words in Chapter 6 of [18]. A generalization to the decomposition $[0,\omega_1)\cup[\omega_1,1)$ is studied in §5 of [5] to deduce an ergodic invariant of the rotation. In this section, we introduce Ostrowski's type numeration system associated to the negative continued fraction of $\xi$ to analyze induced discontinuities. One understands what happens when $\omega_1 \in \mathbb{Z}+\xi\mathbb{Z}$ in [1] from the result of this section.

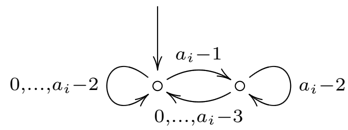
Recall that $a_n$ is the digit of the negative continued fraction of $\xi$ and $Q_n$ is defined by $Q_{n+1} = a_nQ_n - Q_{n-1}$ with $Q_0=0$ and $Q_1=1$.

**Lemma 1.** *Each element $m \in \mathbb{N}$ is uniquely expanded in a form:*

$$m = \sum_{i=1}^{\ell} m_i Q_i$$

*with* $m_i \in \{0, 1, \dots, a_i - 1\}$ *and* $(m_\ell, m_{\ell-1}, \dots, m_1)$ *is a label* [10] *of a finite walk of the next graph:*

(16)




*A word* $(m_\ell, m_{\ell-1}, \dots, m_1) \in \prod_{i=1}^{\ell} \{0, 1, \dots, a_i\}$ *is admissible if this is a label of a walk on this graph. Conversely, for an admissible* $(m_\ell, m_{\ell-1}, \dots, m_1)$, $m = \sum_{i=1}^{\ell} m_i Q_i$ *gives a greedy expansion of* $m$.

*Proof.* The expansion of $m \in \mathbb{N}$ is computed by a greedy algorithm;

(1) $x \leftarrow m$
(2) Find $k \geq 1$ with $Q_k \leq x < Q_{k+1}$
(3) $m_k \leftarrow \lfloor x/Q_k \rfloor$
(4) $x \leftarrow x - m_k Q_k$
(5) If $x = 0$ then stop. Otherwise go back to (2).

We find an expression $m = \sum_{i=1}^{\ell} m_i Q_i$ with $m_i \in \{0, \dots, a_i - 1\}$. Let us prove that this expression is admissible. Reading the graph (16), a word $(m_\ell, m_{\ell-1}, \dots, m_1) \in \prod_{i=1}^{\ell} \{0, 1, \dots, a_i\}$ is admissible if and only if there are no forbidden subwords of the form:

$$(a_p - 1, a_{p-1} - 2, \dots, a_{q+1} - 2, a_q - 1).$$

To be precise, $(m_\ell, m_{\ell-1}, \dots, m_1)$ is admissible if and only if it has no suffix of the form:

$$(a_p - 1, a_{p-1} - 2, \dots, a_{q+1} - 2, a_q - 1, m_{q-1}, \dots, m_1).$$

One can easily prove an equality:

$$(a_p - 1)Q_p + (a_q - 1)Q_q + \sum_{i=q+1}^{p-1} (a_i - 2)Q_i = Q_{p+1} + Q_{q-1} \tag{17}$$

for $1 \leq q < p$. Therefore the words of forbidden form do not appear by the greedy algorithm. This proves that above obtained expansion is admissible. To prove the remaining part, it suffices to show if $m = \sum_{i=1}^{\ell} m_i Q_i$ is admissible, then $m - \sum_{i=k+1}^{\ell} m_i Q_i = \sum_{i=1}^{k} m_i Q_i < Q_{k+1}$, since this means that the admissible expansion must coincides with the greedy one. The digits vector $(m_i, \dots, m_{j+1}, m_j)$ is an *admissible block* if $m_j < a_j - 2$ ($m_j \leq a_j - 2$ for $j = 1$ or $i = j$) and for $k \in \mathbb{N}$ with $j < k < i$ we have $m_k = a_k - 2$ and $m_i = a_i - 1$. In other words, we cut the digit vector into blocks whenever we come back to the left vertex of (16). Note that a length one vector $(m_j)$ with $m_j \leq a_j - 2$ is an admissible block by this definition. The admissibility of $(m_k, m_{k-1}, \dots, m_1)$ allows us to decompose this digits vector into admissible blocks: $(m_{k_1 - 1}, \dots, m_{k_2}), (m_{k_2 - 1}, \dots, m_{k_3}), \dots, (m_{k_v - 1}, \dots, m_1)$ with $k_1 = k + 1$ and $k_{v+1} = 1$. By (17),

$$m_{k_i - 1} Q_{k_i - 1} + \cdots + m_{k_{i+1}} Q_{k_{i+1}} \leq Q_{k_i} + Q_{k_{i+1} - 1} - 2Q_{k_{i+1}}$$

[10] Read from left to right. Each move in (16) decreases the index $i$ by one.

for $i > v$ and the last $-2Q_{k_{i+1}}$ is substituted with $-Q_{k_{i+1}}$ for $i = v$. This inequality is valid for the block of length one with $k_i = k_{i+1} + 1$. Summing up we have

$$\sum_{i=1}^{k} m_i Q_i \le Q_{k+1} - Q_{k_2} + Q_{k_2-1} - Q_{k_3} + Q_{k_3-1} - \cdots - Q_1 + Q_0 < Q_{k+1}$$

which shows the result. □

For a given $x \in [0,1)$, we have another expansion:

$$x = \sum_{i=1}^{\infty} x_i \xi_i$$

with $x_i \in \{0, 1, \dots, a_i - 1\}$. In this case, since $0 < \xi_{n+1} < \xi_n$, the algorithm works in the opposite direction[11], that first we subtract $x_1\xi_1$ and then $x_2\xi_2$ and so on by the greedy choice of digits $x_1, x_2, \dots$. The fact that $Q_i$ and $\xi_i$ satisfies the recurrence of the same shape with symmetric forbidden words, the infinite vector $(x_1, x_2, \dots)$ is a label of the infinite walk on the graph (16). An infinite vector $(x_1, x_2, \dots) \in \prod_{i=1}^{\infty}\{0, 1, \dots, a_i - 1\}$ is *admissible* if it is a label of an infinite walk of the graph (16) which visit the left vertex infinitely many times.

**Lemma 2.** *Each element $x > 0$ is uniquely expanded in a form:*

$$x = \sum_{i=1}^{\infty} x_i \xi_i$$

*with $x_i \in \{0, 1, \dots, a_i - 1\}$ and $(x_1, x_2, \dots)$ is admissible. Conversely, for an admissible vector $(x_1, x_2, \dots)$, $\sum_{i=1}^{\ell} x_i\xi_i$ is a greedy expansion in the above sense.*

It is interesting that the numeration system works by the same greedy algorithm but in two different directions. We call this expansion of $x$, the *dual Ostrowski expansion.*

*Proof.* The expansion of $x$ is computed by a greedy algorithm;

(1) $i \leftarrow 1$
(2) $x_i \leftarrow \lfloor x/\xi_i \rfloor$
(3) $x \leftarrow x - x_i\xi_i$
(4) $i \leftarrow i + 1$ and return to (1).

Clearly $x_i \in \{0, \dots, a_i - 1\}$. Let us show that for an expression $x = \sum_{i=1}^{\infty} x_i\xi_i$, $(x_1, x_2, \dots)$ is admissible. Reading the graph (16), a word $(x_1, x_2, \dots)$ is admissible if and only if there are no subwords of the form:

$$(x_p, \dots, x_q) = (a_p - 1, a_{p+1} - 2, \dots, a_{q-1} - 2, a_q - 1)$$

with $q > p$ and no suffix of the form

$$(x_p, x_{p+1}, \dots) = (a_p - 1, a_{p+1} - 2, a_{p+2} - 2, \dots).$$

We start with an equality analogous to (17):

$$(a_p - 1)\xi_p + (a_q - 1)\xi_q + \sum_{i=p+1}^{q-1} (a_i - 2)\xi_i = \xi_{p-1} + \xi_{q+1} \tag{18}$$

[11] In this case, transition to the next vertex in (16) increases the index $i$ by one.

for $p < q$ which proves that the above expansion contains no words of the form $(a_p - 1, a_{p+1} - 2, \dots, a_{q-1} - 2, a_q - 1)$. Taking $q \to \infty$ in (18), we have

$$(a_p - 1)\xi_p + \sum_{i=p+1}^{\infty} (a_i - 2)\xi_i = \xi_{p-1}$$

which shows the there are no suffix of the form $(a_p - 1, a_{p-1} - 2, a_{p-2} - 2, \dots)$. Thus we have shown that $(x_1, x_2, \dots)$ is admissible.

We will show that if $(x_1, x_2, \dots)$ is admissible, then $x - \sum_{i=1}^{k} x_i\xi_i = \sum_{i=k+1}^{\infty} x_i\xi_i < \xi_k$, since this means that the admissible expansion must coincides with the greedy one. The digits vector $(x_i, \dots, x_j)$ is a *dual admissible block* if $x_j < a_j - 2$ ($x_j \le a_j - 2$ for $i = j$) and for $k \in \mathbb{N}$ with $i < k < j$ we have $x_k = a_k - 2$ and $x_i = a_i - 1$. In short, from the opposite direction of Lemma 1, we cut the digit vector into blocks whenever we come to the left vertex of (16). The admissibility of $(x_{k+1}, x_{k+2}, \dots)$ allows us to decompose this digits vector into dual admissible blocks: $(x_{k_1}, \dots, x_{k_2-1}), (x_{k_2}, \dots, x_{k_3-1}), \dots, (x_{k_v}, \dots, x_{k_v-1}), \dots$ with $k_1 = k+1$. An infinite sequence $(x_{k+1}, x_{k+2}, \dots)$ can not stay eventually in the right vertex of (16), the dual admissible block decomposition is always possible. By (17),

$$x_{k_i}\xi_{k_i} + \dots + x_{k_{i+1}-1}\xi_{k_{i+1}-1} \le \xi_{k_i-1} + \xi_{k_{i+1}} - 2\xi_{k_{i+1}-1}$$

for each $i$. This inequality is valid for the block of length one with $k_{i+1} = k_i + 1$. Therefore we have

$$\sum_{i=k+1}^{\infty} x_i\xi_i \le \xi_{k_1-1} - \xi_{k_2-1} + \xi_{k_2} - \xi_{k_3-1} + \xi_{k_3} - \dots < \xi_k.$$

□

For the proof of the remaining direction of Theorem 2 in §10, it is important to understand how the set of discontinuities vary when the $n$-th rotation in $[0, \xi_n)$ is induced to the $(n+1)$-th. Let $\omega[0]$ be one of the discontinuities of the original rotation $x \mapsto x + \xi$ in $[0, 1)$. Then the recurrence

$$\omega[k] = \omega[k-1] - \left\lfloor \frac{\omega[k-1]}{\xi_k} \right\rfloor \xi_k \tag{19}$$

gives a point of discontinuity of $k$-th induced rotation $x \mapsto x + \xi_{k+1}$ in $[0, \xi_k)$ arose from $\omega[0]$.

**Proposition 4.** *Let $\omega[0] = \sum_{i=1}^{\infty} x_i\xi_i$ be the dual Ostrowski expansion associated with $\xi$. Then we have*

$$\omega[k] = \sum_{i=k+1}^{\infty} m_i\xi_i$$

*Proof.* This is obvious from the greedy algorithm of the dual Ostrowski expansion. □

The point of discontinuity of the form $m\xi \pmod 1$ with $m \in \mathbb{Z}$ plays an exceptional role; for positive $m$ it eventually disappears, and for negative $m$ it eventually corresponds to the point $\xi_k - \xi_{k+1}$:

**Proposition 5.** *Let $m \in \mathbb{Z}$ and $\omega[0] = m\xi \pmod 1$. If $m \geq 0$, then the Ostrowski expansion $m = \sum_{i=1}^{\ell} m_i Q_i$ gives rise to a finite dual Ostrowski expansion*

$$m\xi \pmod 1 = \sum_{i=1}^{\ell} m_i \xi_i.$$

*Thus $\omega[k] = \sum_{i=k+1}^{\ell} m_i \xi_i$ and especially $\omega[\ell] = 0$. If $m < 0$, then we have an infinite dual Ostrowski expansion*

$$m\xi \pmod 1 = \sum_{i=1}^{\infty} x_i \xi_i$$

*and $x_i = a_i - 2$ for sufficiently large $i$. Especially, $\omega[k] = \sum_{i=k+1}^{\infty}(a_i - 2)\xi_i = \xi_k - \xi_{k+1}$ for sufficiently large $k$. Conversely, if $x > 0$ has an expansion $x = \sum_{i=1}^{\infty} x_i \xi_i$ with $x_i = a_i - 2$ for sufficiently large $i$, then $x = m\xi \pmod 1$ with a negative integer $m$.*

*Proof.* The case $m \geq 0$ follows from Lemma 1 and 2 since finite forbidden words for two greedy expansions coincide. By this correspondence of Ostrowski expansions and their duals, the positive $x$'s having finite dual Ostrowski expansions are characterized.

We prove by induction that for any integer $m \geq 1$ there exists $k \geq 0$ such that its dual Ostrowski expansion has a form:

$$-m\xi \pmod 1 = \sum_{i=1}^{\infty} x_i \xi_i$$

with $x_i = a_i - 2$ for $i > k$. The equality (18) implies $\sum_{i=k+1}^{\infty}(a_i - 2)\xi_i = \xi_k - \xi_{k+1}$ for any integer $k \geq 0$. Taking $k = 0$, we have

$$-\xi \pmod 1 = \sum_{i=1}^{\infty}(a_i - 2)\xi_i \tag{20}$$

which shows the case $m = 1$. Assume that

$$-m\xi \pmod 1 = \sum_{i=1}^{\infty} x_i \xi_i$$

with $x_i = a_i - 2$ for $i > k$ and $(x_1, x_2, \dots)$ is admissible. Take the largest integer $s \geq 0$ such that $x_i = 0$ for $i \leq s$. If $s = 0$, then $x_1 > 0$ and $(x_1 - 1, x_2, \dots)$ is admissible. This implies that $-(m+1)\xi \pmod 1$ has the required form. If $s > 0$, then we use the relation (18) in a form

$$(a_1 - 1)\xi_1 + (a_s - 1)\xi_s + \sum_{i=2}^{s-1}(a_i - 2)\xi_i = 1 + \xi_{s+1}. \tag{21}$$

Thus we have

$$\begin{aligned} -(m+1)\xi &= -\xi + \sum_{i=s+1}^{\infty} x_i \xi_i \\ &= \sum_{i=1}^{s-1}(a_i - 2)\xi_i + (a_s - 1)\xi_s + (x_{s+1} - 1)\xi_{s+1} + \sum_{i=s+2}^{\infty} x_i \xi_i \quad \text{in } \mathbb{T} \end{aligned}$$

and $(a_1-2,\dots,a_{s-2}-2,a_{s-1}-1,x_{s+1}-1,x_{s+2},x_{s+3},\dots)$ is easily shown to be admissible. Conversely assume that $x>0$ has an expansion $x=\sum_{i=1}^{\infty} x_i\xi_i$ with $x_i=a_i-2$ for $i>k$ and choose a minimum $k$. If $k=0$, then (20) shows that $x=-\xi$. If $k>0$, then $x_k\le a_k-3$ since $(x_1,x_2,\dots)$ is admissible.

$$\begin{aligned}\sum_{i=1}^{\infty} x_i\xi_i &= \xi_k-\xi_{k+1}+\sum_{i=1}^{k} x_i\xi_i\\ &= \left(Q_k-Q_{k+1}+\sum_{i=1}^{k} x_iQ_i\right)\xi \pmod 1.\end{aligned}$$

By the admissibility of $(x_1,\dots,x_{k-1},x_k+1)$, we have

$$-Q_{k+1}+(x_k+1)Q_k+\sum_{i=1}^{k-1} x_iQ_i<Q_k-Q_{k+1}<0$$

by Lemma 1. □

Define $\omega'[i]$ $(i=0,1,\dots)$ by the same recurrence (19) as $\omega[i]$ with an initial value $\omega'[0]$. Using the dual Ostrowski expansion, one can discuss when two points of discontinuity coincide in the $k$-th induced rotation:

**Proposition 6.** *Assume that $\omega[0]\ne m\xi \pmod 1$ for all $m\in\mathbb{Z}$ and there is an integer $u$ such that $\omega'[0]-\omega[0]=u\xi$ in $\mathbb{T}$. Then there is an integer $k>0$ such that $\omega'[k]=\omega[k]$. Conversely if there exist an integer $k>0$ and $\omega'[k]=\omega[k]$, then there exists an integer $u$ such that $\omega'[0]-\omega[0]=u\xi$ in $\mathbb{T}$.*

The assumption $\omega[0]\ne m\xi \pmod 1$ is necessary. Consider the case $\omega[0]=0$ and $\omega'[0]=1-\xi=-\xi \pmod 1$ in Proposition 5.

*Proof.* In view of Proposition 4 and $\xi_i=Q_i\xi \pmod 1$, the later assertion is obvious. Assume that $\omega'[0]-\omega[0]=u\xi$ in $\mathbb{T}$ for an integer $u$. Without loss of generality, we may assume that $u>0$. By the assumption and Proposition 5, both $\omega[0]$ and $\omega'[0]$ have infinite dual Ostrowski expansions $\omega[0]=\sum_{i=1}^{\infty} x_i\xi_i$ and $\omega'[0]=\sum_{i=1}^{\infty} x_i'\xi_i$ and both of them do not have a suffix of the form $(a_p-2,a_{p+1}-2,\dots)$. We wish to show that there exist $k>0$ such that $x_i=x_i'$ for $i>k$. It suffices to show this in the case $u=1$ since we can iterate $u$-times the procedure. Therefore our goal is to show that $\xi_1+\sum_{i=1}^{\infty} x_i\xi_i$ can be rewritten into $\sum_{i=1}^{\infty} x_i'\xi_i$ without infinite carries. Let $k_1=1$ and $(x_{k_1},\dots,x_{k_2-1}),(x_{k_2},\dots,x_{k_3-2}),\dots$ is the dual admissible block decomposition as in the proof of Lemma 2. When $(x_1+1,x_2,x_3,\dots)$ is admissible, we have nothing to prove. If not, then $x_1+1=a_1$ or $x_1+1=a_1-1$.

*Case* $x_1=a_1-1$. By the admissibility of $(x_1,x_2,\dots)$ the first dual admissible block is of length greater than one, i.e., $k_2>2$. Since $a_1\xi_1=1+\xi_2$, we have

$$a_1\xi_1+\sum_{i=2}^{\infty} x_i\xi_i=(x_2+1)\xi_2+\sum_{i=2}^{\infty} x_i\xi_i \quad \text{in } \mathbb{T}.$$

If $k_2\ge 4$ then $x_2+1=a_2-1$ and we have a new dual admissible block decomposition

$$(0),(x_2+1,x_3,\dots,x_{k_2-1}),(x_{k_2},\dots,x_{k_3-2}),\dots$$

which shows that $x_i=x_i'$ for $i\ge 3$. If $k_2=3$ then $x_2\le a_2-3$. In this case, the same computation gives a new decomposition

$$(0),(x_2+1),(x_{k_2},\dots,x_{k_3-2}),\dots$$

which shows the same $x_i = x'_i$ for $i \geq 3$.

*Case* $x_1 = a_1 - 2$. By the assumption on $(x_1, x_2, \dots)$, there exists a maximum integer $s \geq 2$ such that $x_i = a_i - 2$ for $i < s$. If $x_s \leq a_i - 3$ then $(x_1+1, x_2, \dots, x_{s-1}, x_s)$ forms the first dual admissible block and $(x_1 + 1, x_2, \dots)$ is admissible. This was excluded at the beginning. Therefore we must have $x_s = a_i - 1$, which is the beginning of an dual admissible block $(x_s, \dots, x_\ell)$ with $\ell > s$. Using (21), we have

$$(x_1 + 1)\xi + \sum_{i=2}^{\infty} x_i \xi_i = (x_{s+1} + 1)\xi_{s+1} + \sum_{i=s+2}^{\infty} x_i \xi_i \quad \text{in } \mathbb{T}.$$

If $\ell \geq s + 2$ then $x_{s+1} = a_{s+1} - 2$ and the last expression gives rise to a dual admissible block decomposition:

$$(0), \dots, (0), ((a_{s+1} - 1), x_{s+2}, \dots, x_\ell), \dots$$

which shows that $x_i = x'_i$ for $i \geq s + 2$. If $\ell = s + 1$ then $x_{s+1} \leq a_{s+1} - 3$ and the decomposition is:

$$(0), \dots, (0), (x_{s+1} + 1), \dots$$

and we also have $x_i = x'_i$ for $i \geq s + 2$. □

## 10. Primitive substitutive rotation words are quadratic

We prove that if the general rotation words is primitive substitutive, then $\xi$ is quadratic and $\omega_i - \mu \in \mathbb{Q}(\xi)$, and complete the proof of Theorem 2.

Let $z = z_0 z_1 \cdots \in \mathcal{A}^{\mathbb{N}}$ be *uniformly recurrent*, i.e., for each factor $v$ of $z$ there exists $k > 0$ that all factor of length $k$ must contain $v$ as a factor. Clearly, each general rotation word is uniformly recurrent. Take a prefix $u = z_1 \dots z_n$. One can decompose $z$ into blocks as $z = x_0 x_1 \dots$ with $x_i \in \bigcup_{i=1}^{k} \mathcal{A}^k$ such that $u$ is a prefix of all $x_i x_{i+1} \dots$ and all the factor $u$ of $z$ appear as a prefix of $x_i x_{i+1} \dots$ for some $i$. Such $x_i$ is called the *return words* of $u$. Put $u =^{\#} \{x_i \mid i = 0, 1, \dots\}$ and define a map $\Lambda_{z,u}$ from $\{x_i \mid i = 0, 1, \dots\}$ to $\{0, 1, \dots, u - 1\}$ by the rank of first occurrence of $x_i$ in $z$. The word $\Lambda_{z,u}(x_0)\Lambda_{z,u}(x_1)\Lambda_{z,u}(x_2) \dots$ is the *derived word* of $z$ by the prefix $u$. Then

**Theorem 4** ([14], [19])**.** *A uniformly recurrent word $z$ is primitive substitutive if and only if its derived words by all prefixes form a finite set.*

Note that once this theorem is established, the map $\Lambda_{z,u}$ could be replaced by *any* bijection from $\{x_i \mid i = 0, 1, \dots\}$ to $\{0, 1, \dots, u-1\}$. This change would at most $M!$ times enlarge the number of different derived words, where $M$ is the maximum of the cardinality of return words with respect to a prefix of $z$.

Consider a general rotation word $z = z_0 z_1 \dots$ with respect to a decomposition (1) with an angle $\xi \in [0, 1) \setminus \mathbb{Q}$ and assume that $z$ is primitive substitutive. Take a sufficiently large positive integer $n_0$, so that for $n \geq n_0$, the presumptions of Proposition 3, Proposition 5, Theorem 3 and Proposition 6 are fulfilled, that is,

- Each long word of length $Q_n$ contains two different letters.
- Decomposition into $\mathcal{B}_n$ is unique.
- If $\omega_i^{(n)} - \omega_j^{(n)} = u\xi \pmod 1$ with $i \neq j$ for some integer $u$ then $\{\omega_i^{(n)}, \omega_j^{(n)}\} \subset \{0, \xi_n - \xi_{n+1}, \xi_n\}$.

10.1. **The case** $\mu = 0$**.** First let us show that $\xi$ is quadratic. Choose $m \geq n_0$. From Proposition 3, an occurrence of $z_i \dots z_{i+Q_{m+1}-1} = J_m(0)$ is equivalent to $i\xi \pmod 1 \in [0, \omega_1^{(m)})$. Find a unique $n \geq m$ such that $\xi_{n+1} < \omega_1^{(m)} < \xi_n$, which is always possible under our assumption. We write $n = n(m)$ when it is necessary. Let $z = z_0^{(n)} z_1^{(n)} \dots$ with $z_i^{(n)} \in \mathcal{B}_n$ be the $n$-th renewed word by Theorem 1. Define a map $f_n : \{z_i^{(n)} \mid i = 0, 1, \dots\} \to \{0, 1, \dots, k_n\}$ with $^{\#}\mathcal{B}_n = k_n + 1$ having the order preserving property

$$x \leq_{\text{lex}} y \Longrightarrow f_n(x) \leq f_n(y).$$

Then by construction, $f_n(z_0^{(n)}) f_n(z_1^{(n)}) \dots$ is a coding of the rotation $x \mapsto x + \xi_{n+1}$ of the torus $[0, \xi_n)$ with respect to the decomposition

$$0 = \omega_0^{(n)} < \omega_1^{(n)} < \dots < \omega_{k_n}^{(n)} < \omega_{k_n+1}^{(n)} = \xi_n.$$

There is an index $1 \leq q \leq k_n$ such that $\omega_1^{(m)} = \omega_q^{(n)}$. Hence in this setting, the return word of $z$ with respect to the prefix $J_m(0)$ is a coding of the induced system $[0, \omega_1^{(m)})$ of the $n$-th rotation $x \to x + \xi_{n+1}$ on $[0, \xi_n)$. The first return map $g_m : [0, \omega_1^{(m)}) \to [0, \omega_1^{(m)})$ becomes an exchange of three intervals, which is explicitly given as follows:

$$g_m(x) = \begin{cases} x + \xi_{n+1}, & x \in [0, \omega_1^{(m)} - \xi_{n+1}) \\ x + (b_m + 1)\xi_{n+1} - \xi_n, & x \in [\omega_1^{(m)} - \xi_{n+1}, \xi_n - b_m \xi_{n+1}) \\ x + b_m \xi_{n+1} - \xi_n, & x \in [\xi_n - b_m \xi_{n+1}, \omega_1^{(m)}) \end{cases}$$

with $b_m = \lceil (\xi_n - \omega_1^{(m)})/\xi_{n+1} \rceil$. Define a natural coding $\mathcal{J}_m : [0, \omega_1^{(m)}) \to \mathcal{B}_n^*$ by:

$$\mathcal{J}_m(x) = \begin{cases} J_n(x), & x \in [0, \omega_1^{(m)} - \xi_{n+1}) \\ J_n(x) J_n(x + \xi_{n+1}) \dots J_n(x + b_m \xi_{n+1}), & x \in [\omega_1^{(m)} - \xi_{n+1}, \xi_n - b_m \xi_{n+1}) \\ J_n(x) J_n(x + \xi_{n+1}) \dots J_n(x + (b_m - 1)\xi_{n+1}), & x \in [\xi_n - b_m \xi_{n+1}, \omega_1^{(m)}). \end{cases}$$

The discontinuities $\{\omega_i^{(n)} \mid i = 0, 1, \dots, k_n + 1\}$ are transformed into discontinuities $\{\eta_i^{(m)} \mid i = 0, 1, \dots, m' + 1\}$ of $\mathcal{J}_m$:

$$0 = \eta_0^{(m)} < \eta_1^{(m)} < \dots < \eta_{m'+1}^{(m)} = \omega_1^{(m)}.$$

Obviously we have

$$\{\eta_i^{(m)}\} \cap [0, \omega_1^{(m)} - \xi_{n+1}) = \{\omega_i^{(n)}\} \cap [0, \omega_1^{(m)} - \xi_{n+1})$$

and other elements are of the form $\omega_j^{(n)} - p\xi_{n+1} \in [\omega_1^{(m)} - \xi_{n+1}, \omega_1^{(m)})$ with an integer $p$ with $0 \leq p \leq b_m$. By the assumption, the correspondence between $\{\eta_i^{(m)}\}$ and $\{\omega_i^{(n)}\}$ is one to one and we have $m' = k_n$. Each $\mathcal{J}_m(x)$ with $x \in [0, \omega_1^{(m)})$ is a return word having the prefix $J_m(0)$ and the decomposition of $z$ into return words is written as

$$z = \mathcal{J}_m(0) \mathcal{J}_m(g_m(0)) \mathcal{J}_m(g_m^2(0)) \dots$$

which clearly gives a dynamical interpretation of the return word. The discontinuities of the interval exchange $g_m$ satisfies the *i.d.o.c.* (infinite distinct orbit condition) of [24]. In fact, the denseness of $g_m$-orbits of each $\eta_i^{(m)}$ follows from the minimality of the irrational rotation $x \mapsto x + \xi_{n+1}$ on $[0, \xi_n)$. The assumption $m \geq n_0$ guarantees that two negative orbits of $\omega_i^{(n)}$ and $\omega_j^{(n)}$ with $1 \leq i < j \leq k_n$

do not intersect in the irrational rotation $x \mapsto x + \xi_{n+1}$. This implies that no two negative orbits by $g_m$ of discontinuities $\{\eta_i^{(m)}\}$ intersect. Note that in the case $\omega_1^{(m)} = \xi_n - \xi_{n+1}$, three interval exchange $g_m$ is degenerated into two interval exchange; the irrational rotation. Anyway $g_m$ satisfies i.d.o.c. and therefore $([0, \omega_1^{(m)}), g_m)$ is minimal and uniquely ergodic (see[12] [25], [17], §5.4.1 in [6]). As a result, 1-dimensional Lebesgue measure is the unique invariant measure of $g_m$.

In what follows, we fix a bijection $\Lambda_m$ from $\mathcal{J}_m([0, \omega_1^{(m)}))$ to $\{0, 1, \dots, k_n\}$ by $\Lambda_m(x) = i$ for $x \in [\eta_i^{(m)}, \eta_{i+1}^{(m)})$. Then the derived word of $z$ by the prefix $J_m(0)$ is

$$\Lambda_m(\mathcal{J}_m(0))\Lambda_m(\mathcal{J}_m(g_m(0)))\Lambda_m(\mathcal{J}_m(g_m^2(0))) \cdots \in \{0, 1, \dots, k_n\}^{\mathbb{N}}.$$

By the definition of $\Lambda_m$, we have

$$0 \le x \le y < \omega_1^{(m)} \Longrightarrow \Lambda_m(\mathcal{J}_m(x)) \le_{\text{lex}} \Lambda_m(\mathcal{J}_m(y)).$$

Note that such a derived word is uniquely determined for each $m \ge n_0$. We claim that from a derived word, the value $\xi_{n(m)+1}/\omega_1^{(m)}$ is retrieved uniquely. From a derived word, Theorem 3 and unique ergodicity of $g_m$ allow us to recover the set:

$$\left\{ \frac{\eta_{i+1}^{(m)} - \eta_i^{(m)}}{\omega_1^m} \;\middle|\; i = 0, \dots, k_n \right\}$$

by computing frequencies of letters in $\{0, 1, \dots, k_n\}$.[13] We have to know which letters correspond to the interval $[0, \omega_1^{(m)} - \xi_{n+1})$. By the definition of $\Lambda_m$, these letters should be consecutive integers including 0. Let us denote this set by $\{0, \dots, r\}$. From dynamics of the three interval exchange, we observe that in the derived word, there is a successor letter of $r$ larger than a successor letter of $r + 1$. Moreover $r$ is the minimum letter having this property. Therefore one can compute

$$\frac{\xi_{n(m)+1}}{\omega_1^{(m)}} = 1 - \sum_{i=0}^{r} \frac{\eta_{i+1}^{(m)} - \eta_i^{(m)}}{\omega_1^{(m)}}$$

as claimed. This gives a well defined map to $[0, 1)$ from the set of derived words of $z$ with respect to prefixes $J_m(0)$ with $m \ge n_0$. Theorem 4 says that the set $\{\xi_{n(m)+1}/\omega_1^{(m)} \mid m \ge n_0\}$ must be finite. The same technique allows us to retrieve uniquely the value $(\xi_{n(m)} - b_{n(m)}\xi_{n(m)+1})/\omega_1^{(m)}$ from the derived word. In other words, we uniquely retrieve ratios of three intervals from the derived word and the set

$$\left\{ \left( \frac{\xi_{n(m)+1}}{\omega_1^{(m)}}, \frac{\xi_{n(m)} - b_{n(m)}\xi_{n(m)+1}}{\omega_1^{(m)}} \right) \in \mathbb{R}^2 \;\middle|\; m \ge n_0 \right\}$$

is finite. We deduce that

$$\left\{ \frac{\xi_{n(m)}}{\xi_{n(m)+1}} - b_{n(m)} \;\middle|\; m \ge n_0 \right\}$$

[12] This is an unpublished result in the thesis of M. Boshernitzan (due to S. Ferenczi). Unique ergodicity of a minimal three interval exchange follows from the fact that it is an integral automorphism of an irrational rotation with bounded return time (c.f. [20], Chapter 1, §5).

[13] Approximate by continuous functions the characteristic function of an interval and apply a variant of Theorem 6.19 in [36] for measure theoretical dynamical systems.

is also a finite set. One can take two distinct positive integers $m_1, m_2$ such that

$$\frac{\xi_{n(m_1)}}{\xi_{n(m_1)+1}} - b_{n(m_1)} = \frac{\xi_{n(m_2)}}{\xi_{n(m_2)+1}} - b_{n(m_2)}. \tag{22}$$

Therefore

$$\frac{Q_{n(m_1)}\xi - P_{n(m_1)}}{Q_{n(m_1)+1}\xi - P_{n(m_1)+1}} = \frac{Q_{n(m_2)}\xi - P_{n(m_2)}}{Q_{n(m_2)+1}\xi - P_{n(m_1)+2}} + b_{n(m_1)} - b_{n(m_2)}. \tag{23}$$

If

$$\frac{Q_{n(m_1)}}{Q_{n(m_1)+1}} \neq \frac{Q_{n(m_2)}}{Q_{n(m_2)+1}} + b_{n(m_1)} - b_{n(m_2)},$$

then (23) is a quadratic equation of $\xi$. If

$$\frac{Q_{n(m_1)}}{Q_{n(m_1)+1}} = \frac{Q_{n(m_2)}}{Q_{n(m_2)+1}} + b_{n(m_1)} - b_{n(m_2)},$$

then from $0 < Q_n < Q_{n+1}$ we have $b_{n(m_1)} - b_{n(m_2)} = 0$. From (22), this implies that

$$\frac{\xi_{n(m_1)+1}}{\xi_{n(m_1)}} = \frac{\xi_{n(m_2)+1}}{\xi_{n(m_2)}}$$

which shows that $\xi$ has an eventually periodic continued fraction. In both cases, we have proven that $\xi$ is quadratic.

Now we wish to show that $\omega_i \in \mathbb{Q}(\xi)$ for each $i$. There are integers $N_m, M_m$ such that $\omega_1^{(m)} = \omega_i - N_m\xi - M_m$ with $i \in \{-1, 0, 1, \dots, k-1\}$ where we put $\omega_{-1} = 1-\xi$ for simplicity.[14] From the finiteness of $\{\xi_{n(m)+1}/\omega_1^{(m)} \mid m \geq n_0\}$, one can take an increasing sequence $(m_i)_{i=1,2,\dots}$ of integers such that

$$\omega_{m_i}^{(1)} = \omega_v - N_{m_i}\xi - M_{m_i}$$

with a fixed $v \in \{-1, 0, \dots, k-1\}$ and a constant $s > 0$ that

$$s = \frac{Q_{n(m_1)}\xi - P_{n(m_1)}}{\omega_v - N_{m_1}\xi - M_{m_1}} = \frac{Q_{n(m_2)}\xi - P_{n(m_2)}}{\omega_v - N_{m_2}\xi - M_{m_2}} = \dots.$$

This gives

$$s\left((N_{m_j} - N_{m_1})\xi - (M_{m_j} - M_{m_1})\right) = (Q_{n(m_j)} - Q_{n(m_1)})\xi - (P_{n(m_j)} - P_{n(m_1)})$$

for $j = 2, 3, \dots$. Since $N_{m_j} > N_{m_1}$, we have shown that $s \in \mathbb{Q}(\xi)$. This proves that both $\omega_v$ and $\omega_{m_i}^{(1)}$ are in $\mathbb{Q}(\xi)$. Switching to a subsequence corresponding to the same derived word, we may assume that

$$\frac{\eta_{i+1}^{(m_1)} - \eta_i^{(m_1)}}{\omega_1^{(m_1)}} = \frac{\eta_{i+1}^{(m_2)} - \eta_i^{(m_2)}}{\omega_1^{(m_2)}} = \dots$$

for $i = 0, 1, \dots, k_n$, which is equivalent to

$$\frac{\eta_i^{(m_1)}}{\omega_1^{(m_1)}} = \frac{\eta_i^{(m_2)}}{\omega_1^{(m_2)}} = \dots$$

for $i = 1, 2, \dots, k_n$, since $\eta_0^{(m)} = 0$. As $\eta_i^{(m_j)} \equiv \omega_{\kappa(i,j)} - R_{\kappa(i,j)}\xi \pmod 1$, we can choose for each $j$ a vector

$$\left(\omega_{\kappa(1,j)}, \omega_{\kappa(2,j)}, \dots, \omega_{\kappa(k_n,j)}\right)$$

[14] It is possible that $\omega_{-1} = \omega_i$ holds for some $i \geq 0$.

from a finite set $\{\omega_{-1}, \omega_0, \dots, \omega_{k-1}\}^{k_n}$. Thus taking a subsequence again, we may additionally assume that

$$\eta_i^{(m_j)} \equiv \omega_{\kappa(i)} - R_{\kappa(i,j)}\xi \pmod 1$$

where $\kappa(i)$ does not depend on the choice of $j$. Thus there are integers $S_{\kappa(i,j)}$ such that

$$\frac{\omega_{\kappa(i)} - R_{\kappa(i,1)}\xi - S_{\kappa(i,1)}}{\omega_v - N_{m_1}\xi - M_{m_1}} = \frac{\omega_{\kappa(i)} - R_{\kappa(i,2)}\xi - S_{\kappa(i,2)}}{\omega_v - N_{m_2}\xi - M_{m_2}}$$

which shows that $\omega_{\kappa(i)} \in \mathbb{Q}(\xi)$. According to the discussion in §9, we know that each $\omega_j$ is congruent to some $\omega_{\kappa(i)}$ modulo $\xi\mathbb{Z}$ which finishes the proof for $\mu = 0$.

10.2. **The case $\mu \neq 0$.** We shall skip the details in parts where the analogy to the case $\mu = 0$ works. In §4, we gave a covering the torus $[0, 1)$ by induced systems of the shape $[\alpha, \alpha + \xi)$. The value $\alpha$ was carefully chosen in a form $\alpha \equiv \omega_i \pmod \xi$ for some $i$, not to destroy the combinatorial structure of the assertion. To treat the case $\mu \neq 0$, we need a similar but more flexible covering so that every points on the torus correspond to long words. We explain the construction only for $[0, 1)$, i.e., the first step of the recursive construction as in §4.

The original rotation is the interval exchange on $[0, 1)$ given by (9) with discontinuities $\{\omega_i \mid i = 0, 1, \dots, k-1\}$ as general rotation words. Its induced system on $[\alpha, \alpha + \xi)$ with $\alpha \in [0, 1)$ is

$$\begin{cases} x \mapsto x + \xi_2 & \text{if } x \in [\alpha, \alpha + \xi - \xi_2) \\ x \mapsto x + \xi_2 - \xi & \text{if } x \in [\alpha + \xi - \xi_2, \alpha) \end{cases}$$

and discontinuities are written as

$$\alpha = \omega_0^{(1,\alpha)} < \omega_1^{(1,\alpha)} < \dots < \omega_k^{(1,\alpha)} < \omega_{k+1}^{(1,\alpha)} = \alpha + \xi.$$

Each $\omega_i^{(1,\alpha)}$ has either a form $\alpha + \xi - \xi_2$ or $\omega_j - N_j(\alpha)\xi$ where $N_j(\alpha)$ is the smallest non negative integer that $\alpha \le \omega_j - N_j(\alpha)\xi < \alpha + \xi$ with $j \ge 0$. The later discontinuities are step functions on $\alpha$. The interval $[0, \xi - \xi_2)$ already corresponds to long words. Hence if $\mu \in [0, \xi - \xi_2)$ then there is an interval $[\omega_i^{(1)}, \omega_{i+1}^{(1)}) \ni \mu$ which exactly corresponds to a long word $J_1(\mu) = J_1(\omega_i^{(1)})$ in the sense of Proposition 3. However this is not flexible enough for the later purpose. For a general $\mu \in [0, \xi)$, we choose $\alpha$ such that $\mu \in [\alpha, \alpha + \xi - \xi_2)$ and consider an induced system on $[\alpha, \alpha + \xi)$. Then we find consecutive discontinuities $\omega_i^{(1,\alpha)}$ and $\omega_{i+1}^{(1,\alpha)}$ in this system having the property

$$[\omega_i^{(1,\alpha)}, \omega_{i+1}^{(1,\alpha)}) = \bigcap_{i=1}^{a_1} (J^{-1}(s_i) - (i-1)\xi) \tag{24}$$

where $s_1 \dots s_{a_1} = J_0(\mu)J_0(\mu + \xi) \dots J_0(\mu + (a_1 - 1)\xi)$ is a long word for this shifted system. From the shape of the right side of (24), we see that the interval $[\omega_i^{(1,\alpha)}, \omega_{i+1}^{(1,\alpha)})$ does not depend on the choice of $\alpha \in (\mu - \xi + \xi_2, \mu]$. This explains that if $\mu \in [0, \xi)$, then there is a canonical way to associate an interval $[\omega_i^{(1,\alpha)}, \omega_{i+1}^{(1,\alpha)}) \subset [\alpha, \alpha + \xi)$ which exactly correspond to a long word.

Recall that we had constructed a covering of $[0, 1)$ by induced systems $[\alpha, \alpha + \xi)$ with $\alpha \equiv \omega_i \pmod \xi$ in §4. The induced systems gives a decomposition of the rotation word $z$ into blocks of the shape like $b^{-1}\mathcal{B}_1 b$ or $(k-1)\mathcal{B}_1 b^{-1}$ with some $b \in \mathcal{A}^*$. For each $\mu \in [0, 1)$, there is such an $\alpha$ with $\mu \in [\alpha, \alpha + \xi)$. An advantage of this choice of $\alpha$ is that it only requires minor changes of renewed words, and consequently we may use three assumptions at the beginning of §10 as there are.

Shifting $\alpha$ in an appropriate way, we can associate an interval $[\omega_i^{(1,\alpha)}, \omega_{i+1}^{(1,\alpha)})$ containing $\mu$ which exactly correspond to a long word which satisfies (24). One can iterate this procedure recursively to cover the $(n-1)$-th induced system $[\alpha_{n-1}, \alpha_{n-1}+\xi_{n-1})$ by the $n$-th system $[\alpha_n, \alpha_n+\xi_n)$ such that $\mu \in [\alpha_n, \alpha_n+\xi_n-\xi_{n+1})$ for all $n$. Thus we find an interval $[\omega_i^{(n,\alpha_n)}, \omega_{i+1}^{(n,\alpha_n)})$ ending at consecutive discontinuities of the $n$-th induced system on $[\alpha_n, \alpha_n+\xi_n)$ which satisfies:

$$\mu \in [\omega_i^{(n,\alpha_n)}, \omega_{i+1}^{(n,\alpha_n)}) = \bigcap_{i=1}^{Q_{n+1}} \left(J^{-1}(s_i) - (i-1)\xi\right)$$

with the long word $J(\mu)J(\mu+\xi)\dots J(\mu+(Q_{n+1}-1)\xi) = s_1 s_2 \dots s_{Q_{n+1}}$ produced by this system. Note that the choice of $\alpha_n$ does not affect the three assumptions because shifting of $\alpha_n$ are performed within the covering of §4.

Let us prove that $\xi$ is quadratic. Choose $m \ge n_0$ and find a unique $n = n(m)$ and $i = i(m)$ such that $\xi_{n+1} < \omega_{i(m)+1}^{(m,\alpha_m)} - \omega_{i(m)}^{(m,\alpha_m)} < \xi_n$. Write $\omega_i^{(m)} = \omega_{i(m)}^{(m,\alpha)}$ for simplicity. As stated above, we have some freedom of choice of $\alpha_m$ and it is possible to take $\alpha_m = \omega_i^{(m)}$ for simplicity. Following the same argument as §10.1, the return word is a coding of a three interval exchange transform acts on $[\omega_i^{(m)}, \omega_{i+1}^{(m)})$:

$$g_m(x) = \begin{cases} x+\xi_{n+1}, & x \in [\omega_i^{(m)}, \omega_{i+1}^{(m)} - \xi_{n+1}) \\ x+(b_m+1)\xi_{n+1}-\xi_n, & x \in [\omega_{i+1}^{(m)} - \xi_{n+1}, \omega_i^{(m)} + \xi_n - b_m\xi_{n+1}) \\ x+b_m\xi_{n+1}-\xi_n, & x \in [\omega_i^{(m)} + \xi_n - b_m\xi_{n+1}, \omega_{i+1}^{(m)}) \end{cases}$$

with $b_m = \lceil (\xi_n - \omega_{i+1}^{(m)} + \omega_i^{(m)})/\xi_{n+1} \rceil$. The set of discontinuities of $g_m$ is written as $\{\eta_i^{(m)} \mid i = 0, \dots, k_n+1\}$ and each $\eta_i^{(m)}$ has a form $\omega_j - t\xi$ where $j \in \{-1, 0, \dots, k\}$ and $t \in \mathbb{N}$. In the similar manner as §10.1, one can define a natural coding function $\mathcal{J}_m$ and an appropriate ordering among return words and show that the set

$$\left\{ \left( \frac{\xi_{n(m)+1}}{\omega_{i+1}^{(m)} - \omega_i^{(m)}}, \frac{\xi_{n(m)} - b_{n(m)}\xi_{n(m)+1}}{\omega_{i+1}^{(m)} - \omega_i^{(m)}} \right) \in \mathbb{R}^2 \;\middle|\; m \ge n_0 \right\}$$

is finite and therefore $\xi$ is quadratic.

Our final task is to show that $\omega_i - \mu \in \mathbb{Q}(\xi)$ for all $i$. By a proof analogous to §10.1, there exists a sequence $(m_j)_{j=1,2,\dots}$ that $\omega_{i+1}^{(m_j)} - \omega_i^{(m_j)} \in \mathbb{Q}(\xi)$. Further we may assume that derived words by the prefix of length $m_j$ are identical and

$$\frac{\eta_{i+1}^{(m_1)} - \eta_i^{(m_1)}}{\omega_{i+1}^{(m_1)} - \omega_i^{(m_1)}} = \frac{\eta_{i+1}^{(m_2)} - \eta_i^{(m_2)}}{\omega_{i+1}^{(m_2)} - \omega_i^{(m_2)}} = \dots$$

for $i = 0, 1, \dots, k_n$. This implies that up to renormalization, three interval exchanges $([\omega_i^{(m_j)}, \omega_{i+1}^{(m_j)}), g_{m_j})$ with $j = 1, 2\dots$ have exactly the same shape including the relative location of discontinuities of $\mathcal{J}_{m_j}$. Let us call three intervals appear in the above systems as $I_i$ $(i = 1, 2, 3)$ and define $G(x) = i$ for $x \in I_i$. Then we claim that the map $f$ from $x \in [\omega_i^{(m_j)}, \omega_{i+1}^{(m_j)})$ to $\{1, 2, 3\}^{\mathbb{N}}$ defined by

$$f(x) = G(x)G(g_{m_j}(x))G(g_{m_j}^2(x))\dots$$

is injective. Indeed similarly as in the last part of §7, one can show that the intersection of inverse images becomes a single point, using the fact that the three interval exchange satisfies i.d.o.c., and the $g_m$-orbit of each discontinuity $\eta_i^{(m_j)}$ is dense in $[\omega_i^{(m_j)}, \omega_{i+1}^{(m_j)})$.

Since derived words by the prefix of length $m_j$ are identical and $f(\mu)$ is an image of a morphism of the derived word, $f(\mu)$ is independent of $j$. Thus we see that $\mu$ must be located in the same relative position in the interval $[\omega_i^{(m_j)}, \omega_{i+1}^{(m_j)})$, i.e.,

$$\frac{\mu - \omega_i^{(m_1)}}{\omega_{i+1}^{(m_1)} - \omega_i^{(m_1)}} = \frac{\mu - \omega_i^{(m_2)}}{\omega_{i+1}^{(m_2)} - \omega_i^{(m_2)}} = \dots$$

holds. By using the same technique as §10.1, we can show that $\mu - \omega_i^{(m_j)} \in \mathbb{Q}(\xi)$ and $\mu - \eta_i^{(m_j)} \in \mathbb{Q}(\xi)$. This implies that $\omega_i - \mu \in \mathbb{Q}(\xi)$ as desired.

## 11. Open questions

Theorem 1 and 2 are devoted to coding of $k$ interval exchange transforms which are degenerated into two intervals. A natural question is to generalize these result to genuine interval exchange transforms. Ferenczi, Holton and Zamboni showed in [15] and [16] that an i.d.o.c. three interval exchange gives a primitive substitutive system if and only if the parameters are in the same quadratic field. A related result for $k$ intervals is found in [9].

Let us say a word $z = z_0 z_1 \cdots \in \mathcal{A}^{\mathbb{N}}$ is *primitive substitutive in arithmetic progression* (PSAP), if $z_a z_{a+b} z_{a+2b} \dots$ are primitive substitutive for all $a \geq 0$ and $b > 0$. Theorem 2 implies that a primitive substitutive rotation word is PSAP. Can we characterize PSAP words among primitive substitutive words? Durand had shown in Proposition I.6 of [13] that $z_a z_{a+b} z_{a+2b} \dots$ is an image of the morphism of the fixed point of the substitution, not necessarily primitive. Their primitivity seem to be a subtle question.

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