Pisot expansion in self-inducing systems

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A *Pisot number* is the real root > 1 of a polynomial $x^d - c_{d-1}x^{d-1} - \cdots - c_0$ with $c_i \in \mathbb{Z}$, whose other roots are strictly within the unit circle. If $c_0 = \pm 1$, then it is called a *Pisot unit*. Let $(X, \mathbb{B}, \mu, T)$ be a measure theoretical dynamical system. For any subset $Y \in \mathbb{B}$ of positive measure, an induced system $(Y, \mathbb{B}', \mu', T')$ is canonically defined by the first return map $T'(x) = T^{m(x)}(x) \in Y$. The induced system may behave quite differently from the original. But in cases, there is an expansive affine map π such that $(Y, \mathbb{B}', \mu', T')$ and $(X, \mathbb{B}, \mu, T)$ are conjugate through π. Then we say that it has a *self-inducing structure*. An eigenvalue of the affine map (expansion constant) in a self-inducing system becomes a Pisot number, moreover a Pisot unit, in many important examples.

1. **Substitutive dynamical system**. This is introduced as the simplest self-inducing system. Let σ be a primitive substitution on the monoid $\{1, \ldots, k\}^*$ with a fixed point $x \in \{1, \ldots, k\}^{\mathbb{N}}$, i.e., $\sigma(x) = x$. Let s is the shift $s(x_1 x_2 \ldots) = x_2 x_3 \ldots$. Then the closure of s-orbits of x gives a dynamical system (X_σ, σ) which is strictly ergodic. Pisot conjecture reads if σ is an irreducible Pisot substitution, then (X_σ, σ) has purely discrete spectrum. There are two known ways to access this problem. One is to study directly the spectral measure by observing pattern repetitions (see [7], Solomyak [4]). Another is to construct a geometric realization called *Rauzy fractal*, which is conjugate to the translation of a torus (Arnoux-Ito [3]).

2. **Irrational rotation**.

   Let $X = \mathbb{R}/\mathbb{Z} = [0, 1)$ and $T(x) = x + \alpha$ where $\alpha \in [0, 1) \setminus \mathbb{Q}$. Then (X, T) is strictly ergodic. Standard successive induced systems are given as irrational rotations acting on smaller intervals of lengths α_n. We can compute α_n by the continued fraction of α. By Legendre theorem, one of the induced systems of (X, T) has self-inducing structures iff α is quadratic irrational. Gauss theorem tells that α is *reduced* iff the system (X, T) itself has a self-inducing structure. The expansion constant is a quadratic Pisot unit. See Yasutomi [9] and [2] for self-inducing coding of irrational rotation.

3. **Self-similar tiling**.

   Tiling dynamical system is a natural generalization of substitutive system. Let T_i ($i = 1, \ldots k$) be protiles (closure of its interior) in $\mathbb{R}^2$ and J_i be the set discrete translations such that $\mathbb{R}^2$ is covered by $\bigcup T_i + J_i$ without overlaps. Tiling dynamical system $\mathcal{T}$ is defined by the orbit closure of the tiling by translation $\mathbb{R}^2$ with a suitable topology. If we assume that $\mathcal{T}$ is self-similar (self-inducing), then $(\mathcal{T}, \mathbb{R}^2)$ is strictly ergodic. This system is never mixing but weak mixing is possible. It is known than the system is not weakly mixing (i.e. having a non-trivial point spectrum) iff the expansion constant is a complex Pisot number. Under certain coincidence conditions, $(\mathcal{T}, \mathbb{R}^2)$ becomes purely discrete (see [8]). This coincidence is considered to be close to Pisot property.

4. **Piecewise isometry**.

   This problem arose from discretized process of *rational* rotation. Let ζ_p be a p-th root of unity. Consider a piecewise isometry $T : x \to \zeta_p x$ discontinuously acts on the fundamental parallelogram $X = [0, 1] - \overline{\zeta}_p[0, 1]$. If the image by T goes out of X then it is pulled back into X through real direction. Clearly the 2-dim Lebesgue measure is invariant by T. This gives a highly non-ergodic system and it is conjectured that almost all points in X is periodic. We can observe self-inducing structures when $p \leq 10$ and $p = 12$. Expansion constants are Pisot units (see [6, 1]), and become cubic for $p = 7$ and 9.

My questions are rather obscure and possibly belong to a kind of a folklore.

1. **Need more examples. Related questions.**

   We wish to have more example of self-inducing structures. What about interval exchange transforms or different continued fraction algorithms ? It is nice to find such systems related to Diophantine approximation.

2. **Why do Pisot numbers appear in expansion constants?**

   In above interesting examples, the expansion constant is a Pisot number and even more, a Pisot unit. We do not know the essential reason but a hint is found in self-similar tilings. In this case, we have a special distribution of powers of eigenvalues of associated isometry. We do not know much on tiling with infinite rotational symmetry. Its spectrum is conjectured to be mixed.

3. **Number theoretical applications?**

   If the system has self-inducing structure, then we can naturally associate a number theoretical expansion (multiplicative coding). This means the system has both additive and multiplicative structures (c.f. Kamae's number system [5]). In the best case, this gives an algebraic way to construct a natural extension of a given number theoretical algorithm. Under this, we can study (eventually and purely) periodic orbits in detail. In this manner, we wish to create tight connection between ergodic theory and number theory. A sample question: what is the additive coding of Rosen continued fraction?

References

[1] S. Akiyama, H. Brunotte, A. Pethő, and W. Steiner, *Periodicity of certain piecewise affine planar maps*, Tsukuba J. Math. **32** (2008), no. 1, 1–55.

[2] S. Akiyama and M. Shirasaka, *Recursively renewable words and coding of irrational rotations*, J. Math. Soc. Japan **59** (2007), no. 4, 1199–1234.

[3] P. Arnoux and Sh. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), no. 2, 181–207.

[4] M. Hollander and B. Solomyak, *Two-symbol Pisot substitutions have pure discrete spectrum*, Ergodic Theory Dynam. Systems **23** (2003), no. 2, 533–540.

[5] T. Kamae, *Numeration systems as dynamical systems—introduction*, Dynamics & stochastics, IMS Lecture Notes Monogr. Ser., vol. 48, Inst. Math. Statist., Beachwood, OH, 2006, pp. 198–211.

[6] K. Koupstov, J. H. Lowenstein, and F. Vivaldi, *Quadratic rational rotations of the torus and dual lattice maps*, Nonlinearity **15** (2002), 1795–1842.

[7] M. Queffélec, *Substitution dynamical systems—Spectral analysis*, Lecture Notes in Mathematics, vol. 1294, Springer-Verlag, Berlin, 1987.

[8] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 695–738.

[9] S. Yasutomi, *On sturmian sequences which are invariant under some substitutions*, Number theory and its applications (Kyoto, 1997), Kluwer Acad. Publ., Dordrecht, 1999, pp. 347–373.