POSITIVE FINITENESS OF NUMBER SYSTEMS

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Abstract We characterize the set of β's that each polynomial in base β with non-negative integer coefficients has a finite admissible expression in some number systems.

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1. Introduction

In this note, we study a certain finiteness property of number systems given by power series in some base β, which are called *beta-expansion* and *canonical number system*.

In relation to symbolic dynamics, an important problem is to determine the set of β's that each polynomial in base β with non-negative integer coefficients has a finite expression in the corresponding number system. However this problem may be pretty difficult in general. We narrow our scope on the set of such β's which does *not* have 'global' finiteness. Let us explain exactly the problem for beta-expansion (c.f. [27]).

Let $\beta > 1$ be a real number. Each positive x is uniquely expanded into a *beta-expansion*:

$$x = \sum_{i=M}^{\infty} a_i \beta^{-i} \qquad (M \text{ could be negative})$$

under conditions

$$a_i \in [0, \beta) \cap \mathbb{Z} \quad \text{and} \quad \forall L \geq M \quad 0 \leq x - \sum_{i=M}^{L} a_i \beta^{-i} < \beta^{-L},$$

which is also called *greedy expansion*. We write this expression as

$$x = x_M x_{M+1} \dots x_0.x_1 x_2 \dots$$

following an analogy to the usual decimal expansion. If $a_i = 0$ for sufficiently large i, then the expansion is called finite and the tail $00\dots$ can be omitted as usual. Let $\mathrm{Fin}(\beta)$ be the set of finite beta expansions. It is obvious that $\mathrm{Fin}(\beta)$ is a subset of $\mathbb{Z}[1/\beta]\cap[0,\infty)$ if β were an algebraic integer[1]. Frougny and Solomyak [14] firstly studied the property

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap [0,\infty)$$

which we call *finiteness* property (F). If β has the property (F), then β is a Pisot number, that is, a real algebraic integer greater than one that all other conjugates of β have modulus less than one.

A polynomial $x^d - a_{d-1}x^{d-1} - \dots - a_0$ with $a_{d-1} \geq a_{d-2} \geq \dots \geq a_0 > 0$ gives a Pisot number $\beta > 1$ as a root (c.f. [10]). Then in [14] it is shown that the property (F) holds for this class of β. The full characterization of β with (F) among algebraic integers (or among Pisot numbers), is a difficult problem when $d \geq 3$ (c.f. [2], [8], [4]).

The expansion of 1 is a digit sequence given by an expression $1 = \sum_{i=1}^{\infty} c_i \beta^{-i} = .c_1c_2c_3\dots$ such that $.0c_2c_3\dots$ is the beta expansion of $1 - c_1/\beta$ with $c_1 = \lfloor \beta \rfloor$. This expansion play a crucial role to determine which formal expression could be realized as beta-expansion ([25], [18]). Especially a formal expression

$$1 = \sum_{i=1}^{\infty} d_i \beta^{-i} = .d_1 d_2 \dots$$

coincides with the expansion of 1 if and only if the digit sequence $d_1d_2\dots$ is greater than its left shift $d_i d_{i+1}\dots$ for $i > 1$ by the natural lexicographical order.

In [14] it is shown that if the expansion of $1 = .c_1c_2\dots$ has infinite decreasing digits (i.e., $c_1 \geq c_2 \geq c_3 \geq \dots$ and $c_i = c_{i+1} > 0$ from some index on), then the set $\mathrm{Fin}(\beta)$ is closed under addition. This is equivalent to the condition:

$$\mathbb{Z}_+[\beta] \subset \mathrm{Fin}(\beta)$$

where $\mathbb{Z}_+ = \mathbb{Z} \cap [0,\infty)$ and $\mathbb{Z}_+[\beta]$ is the set of polynomials in base β with coefficients in $\mathbb{Z}_+$. We call this property *positive finiteness* ((PF) for short). The author showed in [3] that (PF) implies *weak finiteness*

[1] If β is an algebraic integer, then $\mathbb{Z}[\beta] \subset \mathbb{Z}[1/\beta]$.

which has close connection to Thurston's tiling generated by Pisot unit β (c.f. [30], [8]). One motivation to study (PF) comes from this fact.

In [9], Ambrož, Frougny, Masáková and Pelantová gave a characterization of (PF) in terms of 'transcription' of minimal forbidden factors. Our problem in this paper is to characterize β with the property (PF) *without* (F). By this restriction of the scope, we can give a complete characterization of such β's:

Theorem 1. *Let $\beta > 1$ be a real number with positive finiteness. Then either β satisfies the finiteness property (F) or β is a Pisot number whose minimal polynomial is of the form:*

$$x^d - (1 + \lfloor\beta\rfloor)x^{d-1} + \sum_{i=2}^{d} a_i x^{d-i}$$

with $a_i \geq 0$ $(i = 2, \dots, d)$, $a_d > 0$ and $\sum_{i=2}^{d} a_i < \lfloor\beta\rfloor$. In the later case, the expansion of 1 *has infinite decreasing digits. Conversely if $\beta > 1$ is a root of the polynomial*

$$x^d - Bx^{d-1} + \sum_{i=2}^{d} a_i x^{d-i}$$

with $a_i \geq 0$, $a_d > 0$ and $B > 1 + \sum_{i=2}^{d} a_i$, then this polynomial is irreducible and β is a Pisot number with (PF) without (F). We also have $B = 1 + \lfloor\beta\rfloor$.

The study of (PF) is reduced to that of (F) by Theorem 1. Unfortunately as a result, nothing new exists in (PF) but the ones already found in [14].

A parallel problem is solved in another well known number system. Let α be an algebraic integer [2] of degree d having its absolute norm $|N(\alpha)|$. If each element $x \in \mathbb{Z}[\alpha]$ has an expression:

$$x = \sum_{i=0}^{\ell} a_i \alpha^i, \quad a_i \in \mathcal{A} = \{0, 1, \dots, |N(\alpha)| - 1\}$$

then we say that α gives a *canonical number system* (CNS for short). If such expression exists, then it is unique since $\mathcal{A}$ forms a complete set of representatives of $\mathbb{Z}[\alpha]/\alpha\mathbb{Z}[\alpha]$ and the digit string is computed from

[2] α is used instead of β to distinguish the difference of number systems.

the bottom by successive consideration modulo α. If α gives a CNS, then α must be expanding, that is, all conjugates of α have modulus greater than one ([22]). Assume that α has the minimal polynomial of the form $x^2 + Ax + B$. Then α gives a CNS if and only if $-1 \le A \le B$ and $B \ge 2$ ([19], [20], [15]). When $d \ge 3$, the characterization of α's among expanding algebraic integers is again a difficult question ([6],[28], [7],[11],[12], [5]). It is obvious that CNS is an analogous concept of (F). To pursue this analogy, let us say that α has *positive finiteness* if $\mathbb{Z}_+[\alpha] = \mathcal{A}[\alpha]$, i.e.,

$$\forall 0 \le a_i \in \mathbb{Z} \quad \exists b_j \in \{0, 1, \dots, |N(\alpha)| - 1\} \qquad \sum_{i \ge 0} a_i \alpha^i = \sum_{j \ge 0} b_j \alpha^j.$$

This positive finiteness is in fact weaker than CNS and we can show

Theorem 2. *Assume that α has positive finiteness. Then either α gives a CNS or the minimal polynomial of α is given by*

$$\sum_{i=1}^{d} a_i x^i - C \tag{1}$$

with $a_d = 1$, $a_i \ge 0$ and $\sum_{i=1}^{d} a_i < C$. Conversely if α is a root of the irreducible polynomial (1) with the same condition then α has positive finiteness but does not give a CNS.

It is not possible to remove irreducibility in the last statement. For example, $x^2 + x - 12 = (x - 3)(x + 4)$ but -4 gives a CNS.

In [26], Pethő introduced a more general concept 'CNS polynomial' among expanding polynomials. If the polynomial is irreducible, then the concept coincides with CNS. It is straightforward to generalize above Theorem 2 to this framework. In this extended sense, $x^2 + x - 12$ has positive finiteness.

2. Proof of Theorem 1.

First we prove the later part of the Theorem 1. Assume that $\beta > 1$ is a root of a polynomial:

$$P(x) = x^d - Bx^{d-1} + \sum_{i=2}^{d} a_i x^{d-i} \quad \text{with } a_i \ge 0,\ a_d > 0 \text{ and } B > 1 + \sum_{i=2}^{d} a_i.$$

By applying Rouché's Theorem, $P(x)$ and $x^d - Bx^{d-1}$ has the same number of roots in the open unit disk. Thus β is a Pisot number and $P(x)$ is irreducible. In fact, if $P(x)$ is non trivially decomposed into

$P_1(x)P_2(x)$ and $P_1(\beta) = 0$, then the constant term of $P_2(x)$ is less than 1 in modulus, and hence it must vanishes. This contradicts $a_d > 0$.

The relation $P(\beta) = 0$ formally gives rise to a relation

$$1 = .B\ \overline{a_2}\ \overline{a_3}\ \dots\ \overline{a_d}$$

where we put $\overline{x} = -x$ to simplify the notation. Multiplying β^{-j} $(j = 1, 2, \dots)$ and summing up we have

$$\begin{aligned} 1 &= .B\ \overline{a_2}\ \overline{a_3}\ \dots\ \overline{a_d} \\ &+ .\overline{1}\ B\ \overline{a_2}\ \dots\ \overline{a_{d-1}}\ \overline{a_d} \\ &+ .0\ \overline{1}\ B\ \dots\ \overline{a_{d-2}}\ \overline{a_{d-1}}\ \overline{a_d} \\ &+ \dots . \\ &= .(B-1)\ (B-1-a_2)\ (B-1-a_2-a_3)\ \dots\ m\ m\ m \dots \end{aligned}$$

with $m = B - 1 - \sum_{i=2}^{d} a_i$. As the last sequence is lexicographically greater than its left shifts, this gives the expansion of 1 of β with infinite decreasing digits. By the result of [14], this β has the property (PF). Now it is clear that $B = 1 + \lfloor \beta \rfloor$. As the expansion of 1 is not finite, β does not satisfy (F). This is also shown in the following way. Since $P(0) < 0$ and $P(1) > 0$, there is a positive conjugate $\beta' \in (0, 1)$. Using Proposition 1 of [1], β does not satisfy the finiteness property (F).

To prove the first part, we quote two lemmas.

Lemma 3 (Theorem 5 in Handelman [17]). *Let $\beta > 1$ be an algebraic integer such that other conjugates has modulus less than β and there are no other positive conjugates. Then β is a Perron-Frobenius root of a primitive companion matrix.*

The proof of this lemma relies on the Perron-Frobenius theorem and the fact that for any polynomial $p(x)$ without positive roots, $(1+x)^m p(x)$ have only positive coefficients for sufficiently large m. (A direct proof of this fact will be given in the appendix.) We need another

Lemma 4 (Lemma 2 in [14]). *An equality $\mathbb{Z}_+[\beta] = \mathbb{Z}[\beta] \cap [0, \infty)$ holds if and only if β is a Perron-Frobenius root of a primitive companion matrix.*

In the following, we also use the fact that there are only two Pisot numbers less than $\sqrt{2}$. The smallest one, say $\theta \approx 1.32372$, is a positive root of $x^3 - x - 1$ and the next $\theta_1 \approx 1.38028$ is given by $x^4 - x^3 - 1$ (c.f. [24]). C.L. Siegel [29] firstly proved that they are the smallest two Pisot numbers. In [1], it is shown that θ has property (F). On the other hand, θ_1 does not satisfy (PF) since $\theta_1 + 1$ has the infinite purely periodic beta expansion $100.0010000100001\dots$.

Let us assume that $\beta > 1$ has positive finiteness (PF) but does not have the property (F). This implies that β is not an integer and greater than $\sqrt{2}$. Since $\mathbb{Z}_+ \subset \mathrm{Fin}(\beta)$, Proposition 1 of [2] implies that β is a Pisot number. We claim that β has a conjugate $\beta' \in (0,1)$. If not, then by Lemma 3, β is a Perron-Frobenius root of a primitive companion matrix. Then by Lemma 4, each element of $\mathbb{Z}[\beta] \cap [0,\infty)$ has a polynomial expression in base β with non-negative integer coefficients. Thus (PF) property implies the property (F). This is a contradiction which shows the claim.

By the property (PF), $\kappa = (1 + \lfloor \beta \rfloor)/\beta \in \mathrm{Fin}(\beta)$. Note that $\beta > \sqrt{2}$ implies $\lfloor \beta \rfloor + 1 < \beta^2$ and the beta expansion of κ begins with $a_0 = 1$. Hence, as $\kappa - 1 < \beta^{-1}$, we have a beta expansion:

$$\frac{1 + \lfloor \beta \rfloor}{\beta} = 1.0a_2a_3 \dots a_\ell$$

with $a_\ell \neq 0$. Set $Q(x) = x^\ell - (\lfloor \beta \rfloor + 1)x^{\ell-1} + \sum_{i=2}^{\ell} a_i x^{\ell-i}$. Then $Q(x)$ has two sign changes in its coefficients. By Descartes's law, there exist at most two positive real roots of $Q(x)$, and therefore they must be β and β'. On the other hand, we see $Q(0) = a_\ell > 0$. If $Q(1) > 0$ then there are at least two positive root of $Q(x)$ in $(0,1)$ which is absurd. Thus we have $Q(1) < 0$ which implies $\sum_{k=2}^{\ell} a_k < \lfloor \beta \rfloor$. We have already proven under this inequality that $Q(x)$ is irreducible and the expansion of 1 of β has infinite decreasing digits. $\square$

A few words should be added to make clear the situation. If $\lfloor \beta \rfloor + 1$ has a finite beta expansion in base β, the above procedure yields the same polynomial $Q(x) = x^\ell - (1 + \lfloor \beta \rfloor)x^{\ell-1} + \sum_{k=2}^{\ell} a_k x^{\ell-k}$. Since $\beta > 1$ is a root of $Q(x)$ and $Q(0) > 0$, $Q(x)$ has exactly two positive real roots. If $Q(1) < 0$, then β has (PF) by the same reasoning. If $Q(1) \geq 0$, then there is a root $\eta \geq 1$ other than β. Note that this could happen even if β has property (PF). However in such case, $Q(x)$ must be reducible since β does not have other positive conjugate if it has property (PF). Especially if β satisfies (F), then $Q(x)$ is reducible. For example, $\beta = (1 + \sqrt{5})/2$ satisfies (F) and $Q(x) = x^3 - 2x^2 + 1 = (x^2 - x - 1)(x - 1)$. The above proof shows, as a consequence, that $Q(x)$ must be irreducible if β satisfies (PF) without (F).

It is not clear whether the condition $\mathbb{Z}_+ \subset \mathrm{Fin}(\beta)$ implies (PF). We have difficulty in proving the existence of a positive conjugate $\beta' \in (0,1)$ under this condition.

3. Proof of Theorem 2.

First we recall that if α has positive finiteness, then α is expanding. This was proved in CNS case in [22] and the same proof works in positive finiteness case. (See Lemma 3 and the proof of Theorem 3 in [22].) [3]

Let us assume that α has positive finiteness but does not give a CNS. Let $P(x)$ be the minimal polynomial of α. We claim that there exists a positive conjugate α'. Suppose not. Then by the remark after Lemma 3, there is a large integer M that $(1+x)^M P(x)$ has only positive coefficients. This gives a relation of the form $\sum_{i=0}^{\ell} a_i\alpha^i = 0$ with $a_i > 0$. Thus each element of $\mathbb{Z}[\alpha]$ has an equivalent expression in $\mathbb{Z}_+[\alpha]$ which is attained by repeated addition of the above relation. This shows that $\mathbb{Z}_+[\alpha] = \mathbb{Z}[\alpha]$ and positive finiteness of α implies that α gives a CNS. This is a contradiction and the claim is proved. Note that $\alpha' > 1$.

Let $C = |N(\alpha)|$ and write its expression $C = \sum_{i=0}^{d} a_i\alpha^i$ with $a_i \in \mathcal{A}$. Taking modulo α, we see that $a_0 = 0$. Set $Q(x) = \sum_{i=1}^{d} a_i x^i - C$. As $Q(0) < 0$ and there is only one sign change in the coefficients of $Q(x)$, there exists exactly one positive root of $Q(x)$ which is α'. Now $\alpha' > 1$ implies $Q(1) < 0$, i.e., $\sum_{i=1}^{d} a_i < C$. Suppose that $Q(x)$ is not irreducible and $Q(x) = P(x)R(x)$ with $\deg R \geq 1$. From $C = |N(\alpha)|$, we deduce $|R(0)| = 1$ and hence there exists a root η of $Q(x)$ with $|\eta| \leq 1$. Then

$$0 = |Q(\eta)| = \left|\sum_{i=1}^{d} a_i\eta^i - C\right| \geq C - \sum_{i=1}^{d} a_i$$

gives a contradiction. This shows that $Q(x) = P(x)$ and $a_d = 1$.

Finally we prove the converse. Assume that α is a root of the irreducible polynomial $Q(x) = \sum_{i=1}^{d} a_i x^i - C$ with $a_d = 1$, $a_i \geq 0$ and $\sum_{i=1}^{d} a_i < C$. Then $Q(x)$ must be expanding since otherwise there exists a root η with $|\eta| \leq 1$ of $Q(x)$ and we shall meet the same contradiction. As $Q(0) < 0$ there exists a positive conjugate α'. Hence α can not give a CNS, since -1 can not have finite expansion (c.f. Proposition 6 in [15]). It remains to show that α has positive finiteness. The idea of this proof can be traced back to [21].

As α is a root of $Q(x)$, we have an expression

$$a_d a_{d-1} \ \ldots \ a_1 \ \overline{C} = 0. \tag{2}$$

We describe an algorithm from each $x = \sum_{i=0}^{\ell} d_i\alpha^i$ with $d_i \in \mathbb{Z}_+$ how to get an equivalent expression in $\mathcal{A}[\beta]$. Adding $\kappa = \lfloor d_0/C \rfloor$ times the

[3]For the later use, it suffices to show an easier fact (Lemma 3 in [22]): 'each conjugate of α has modulus not less than one.'

relation (2), we have an equivalent expression of x in $\mathbb{Z}_+[\alpha]$:

$$d_\ell d_{\ell-1}\dots d_0 + \kappa \times \left(a_d a_{d-1}\ \dots\ a_1\ \overline{C}\right) = d'_{\ell'} d'_{\ell'-1}\dots d'_0$$

whose constant term is $d'_0 = d_0 - \kappa C \in \mathcal{A}$. Repeat the same process on d'_1 to make the coefficients of α^1 into $\mathcal{A}$. This process can be continued in a similar manner. In each step, the sum of digits of the expression of x is strictly decreasing. Hence we finally get an expression in $\mathcal{A}[\alpha]$ in finite steps. □

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Appendix

Handelman showed in [16], as a special case of his wide theory, that for any polynomial $p(x) \in \mathbb{R}[x]$ having no non-negative roots, there exists a positive integer M that $(1+x)^M p(x)$ has only positive coefficients (c.f. [23] and [13]). This is a crucial fact in proving Lemma 3 and Theorem 2. As the statement itself looks elementary, it may be worthy to note here a direct short proof. To prove this we factorize $p(x)$ into quadratic and linear factors in $\mathbb{R}[x]$. Since a linear factor $(x+a)$ with $a > 0$ does no harm, we prove that for any x^2+bx+c with $b^2 < 4c$ there exists a positive n that $(1+x)^n(x^2+bx+c)$ has positive coefficients. The k-th coefficient of $(1+x)^n(x^2+bx+c)$ is

$$\binom{n}{k}\left(c\frac{n-k}{k+1}+b+\frac{k-1}{n-k+1}\right).$$

Thus we show that $f(k) = c(n-k)(n-k+1)+b(k+1)(n-k+1)+(k-1)(k+1) > 0$ for $k = 0, 1, \ldots, n$ if n is suffciently large. From an expression

$$f(k) = -1+b+bn+cn+cn^2+(-c+bn-2cn)k+(1-b+c)k^2,$$

as $x^2+bx+c > 0$ implies $1-b+c > 0$, the minimum of $f(k)$ is attained when $k = (c-bn+2cn)/(2-2b+2c)$. Direct computation shows

$$f(k) \geq \frac{-4+8b-4b^2-4c+4bc-c^2+(4b-4b^2+4c+2bc)n+(-b^2+4c)n^2}{4(1-b+c)}.$$

As the coefficient of n^2 in the numerator is positive, the assertion is shown. □

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