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Pisot number system and its dual tiling

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Abstract. Number systems in Pisot number base are discussed in relation to arithmetic construction of quasi-crystal model. One of the most important ideas is to introduce a 'dual tiling' of this system. This provides us a geometric way to understand the 'algebraic structure' of the above model as well as dynamical understanding of arithmetics algorithms.

Keywords. Pisot Number, Number System, Symbolic Dynamical System, Tiling

1. Beta expansion and Pisot number system

For this section, the reader finds a nice survey by Frougny [41]. However we give a brief review and concise proofs of fundamental results to make this note more self-contained. Let us fix $\beta > 1$ and $\mathcal{A} = [0, \beta) \cap \mathbb{Z}$. Denote by $\mathcal{A}^*$ the set of finite words over $\mathcal{A}$ and by $\mathcal{A}^{\mathbb{N}}$ the set of right infinite words over $\mathcal{A}$. By concatenation $\oplus$:

$$a_1a_2 \dots a_n \oplus b_1b_2 \dots b_m = a_1a_2 \dots a_nb_1b_2 \dots b_m,$$

$\mathcal{A}^*$ forms a monoid with the empty word λ as an identity. An element of $\mathcal{A}^*$ is embedded into $\mathcal{A}^{\mathbb{N}}$ by concatenating infinite $00\dots$ to the right. $\mathcal{A}^{\mathbb{N}}$ becomes a compact metric space by the distance function

$$p(a_1a_2 \dots, b_1b_2 \dots) = 2^{-j}$$

for the smallest index j with $a_j \neq b_j$. A lexicographical order of $\mathcal{A}^{\mathbb{N}}$ is given by $a_1a_2 \dots <_{\text{lex}} b_1b_2 \dots$ if $a_j < b_j$ at the same index j. The shift operator σ acts continuously on $\mathcal{A}^{\mathbb{N}}$ by $\sigma(a_1a_2 \dots) = a_2a_3 \dots$ and the pair $(\mathcal{A}^{\mathbb{N}}, \sigma)$ forms a topological dynamical system, which is called the full shift over $\mathcal{A}$. We shall later need $\mathcal{A}^{\mathbb{Z}}$ the set of bi-infinite words over $\mathcal{A}$. Each element of $\mathcal{A}^{\mathbb{Z}}$ is written as $(a_i)_{i\in\mathbb{Z}} = \dots a_{-1}a_0 \bullet a_1a_2 \dots$ where the symbol $\bullet$ is used as a usual decimal point which indicates the place where the index 1 starts. In this case the metric is given by

$$p(\dots a_{-1}a_0 \bullet a_1a_2 \dots, \dots b_{-1}b_0 \bullet b_1b_2 \dots) = 2^{-j}$$

with the smallest index $j \geq 0$ with $(a_j, a_{-j}) \neq (b_j, b_{-j})$ and the shift is defined by $\sigma((a_n)) = (a_{n+1})$. For both $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$, the closed σ invariant subset is called the *subshift*.

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The *beta transformation* is a piecewise linear map T_β on $[0,1)$ defined by

$$T_\beta : x \longrightarrow \beta x - \lfloor \beta x \rfloor$$

which was shown to be ergodic by Rényi [45]. Parry [42] gave the invariant measure of this system, which is absolutely continuous to the Lebesgue measure and its Radon-Nikodym derivative was made explicit. For each real $x = x_1 \in [0,1)$, iterating beta transforms we have

$$T_\beta : x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} \dots .$$

The label over the arrow is defined as $a_i = \lfloor \beta x_i \rfloor$. One can expand $x \in [0,1)$ into

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \dots$$

and $a_i \in \mathcal{A}$. Denote by $d_\beta : [0,1) \ni x \to a_1 a_2 \dots \in \mathcal{A}^{\mathbb{N}}$. Then d_β is order preserving, that is, $x < y$ implies $d_\beta(x) <_{\mathrm{lex}} d_\beta(y)$. We confirm a commutative diagram:

$$\begin{tikzcd} {[0,1)} \arrow[r, "T_\beta"] \arrow[d, "d_\beta"'] & {[0,1)} \arrow[d, "d_\beta"] \\ \mathcal{A}^{\mathbb{N}} \arrow[r, "\sigma"'] & \mathcal{A}^{\mathbb{N}} \end{tikzcd} \tag{1}$$

Define the realization map:

$$\pi = \pi_\beta : a_1 a_2 \dots \in \mathcal{A}^{\mathbb{N}} \longrightarrow \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \in \mathbb{R}.$$

Note that π_β is continuous but d_β is not. Since $\pi_\beta \circ d_\beta(x) = x$ by definition, we have $\pi_\beta(\mathcal{A}^{\mathbb{N}}) \supset [0,1)$. However $\mathcal{A}^{\mathbb{N}} \not\subset d_\beta([0,1))$. If $a_1 a_2 \dots \in \mathcal{A}^{\mathbb{N}}$ is contained in $d_\beta([0,1))$, we say that $a_1 a_2 \dots \in \mathcal{A}^{\mathbb{N}}$ is *admissible*. A finite word $a_1 a_2 \dots a_m$ of $\mathcal{A}^*$ is admissible if its right completion $a_1 a_2 \dots a_m \oplus 00 \dots \in \mathcal{A}^{\mathbb{N}}$ is admissible. For a given positive x, there is an integer $m > 0$ with $\beta^{-m} x \in [0,1)$, x can be expanded like

$$x = a_{-m}\beta^m + a_{-m+1}\beta^{m-1} + \dots + a_0 + \frac{a_1}{\beta} + \dots$$

This is the *beta expansion* which is a natural generalization of usual decimal or binary expansion. By abuse of terminology [1], we sometimes write

$$d_\beta(x) = a_{-m} a_{-m+1} \dots a_{-1} a_0 \bullet a_1 a_2 a_3 \dots$$

or even simply

[1]The symbol '$\bullet$' is not in $\mathcal{A}$

$$x = a_{-m}a_{-m+1}\dots a_{-1}a_0 \bullet a_1a_2a_3\dots$$

if there is no room of confusion. The expansion is *finite* if there is an ℓ that $a_n = 0$ holds for $n > \ell$ and we denote by

$$x = a_{-m}a_{-m+1}\dots a_0 \bullet a_1a_2a_3\dots a_\ell.$$

Set

$$d_\beta(1-0) = \lim_{\varepsilon \downarrow 0} d_\beta(1-\varepsilon).$$

by the metric of $\mathcal{A}^{\mathbb{N}}$. Then $d_\beta(1-0)$ can not be finite.

Theorem 1 ([42], [39]). *A right infinite word $\omega = \omega_1\omega_2\dots \in \mathcal{A}^{\mathbb{N}}$ is admissible if and only if $\sigma^n(\omega) <_{\mathrm{lex}} d_\beta(1-0)$ holds for all $n = 0, 1, \dots$.*

Proof. Let $d_\beta(1-0) = c_1c_2\dots$. It suffices to show that

$$\pi(\sigma^n(\omega)) = \sum_{i=1}^{\infty} \omega_{n+i}\beta^{-i} \in [0,1).$$

By the assumption, there exists an *admissible block decomposition*:

$$\omega_{n+1}\omega_{n+2}\dots = c_1\dots c_{k_1-1}g_1c_1\dots c_{k_2-1}g_2c_1\dots c_{k_3-1}g_3\dots$$

where $g_i < c_{k_i}$ and $k_i \geq 1$. It is easily seen that

$$\pi(c_1\dots c_{k_i-1}g_i) \leq \pi(c_1\dots c_{k_i-1}(c_{k_i}-1)) < 1 - \frac{1}{\beta^{k_i}}$$

and therefore

$$\pi(\omega_{n+1}\omega_{n+2}\dots) < 1 - \frac{1}{\beta^{k_1}} + \frac{1}{\beta^{k_1}}\left(1 - \frac{1}{\beta^{k_2}}\right) + \frac{1}{\beta^{k_1+k_2}}\left(1 - \frac{1}{\beta^{k_3}}\right) + \dots = 1.$$

$\square$

Take $\mathcal{F} \subset \mathcal{A}^*$ and define a subset $\mathcal{A}_{\mathcal{F}}$ of $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ by the infinite words whose subwords are not in $\mathcal{F}$. Then $\mathcal{A}_{\mathcal{F}}$ is a subshift and any subshift is written in this manner. Thus $\mathcal{F}$ is the set of *forbidden words*. A subshift is called *of finite type* if there is a finite set $\mathcal{F}$ and it is expressed as $\mathcal{A}_{\mathcal{F}}$. A subshift $\mathcal{A}_{\mathcal{F}}$ is called *sofic* if one can choose $\mathcal{F}$ which is recognizable by a finite automaton. A subshift of finite type is sofic and the sofic shift is characterized as a factor of the shift of finite type. A sofic shift $\mathcal{A}_{\mathcal{F}}$ is nothing but the set of infinite labels which is generated by infinite walks on a fixed finite directed graph labelled by $\mathcal{A}$ (c.f. [40]).

The *beta shift* X_β is a subshift of $\mathcal{A}^{\mathbb{Z}}$ which is defined to be a set of bi-infinite words whose all finite subwords are admissible. X_β is sofic if and only if $d_\beta(1-0)$ is eventually periodic. Such a β is designated as a *Parry number*. Further $d_\beta(1-0)$ is purely periodic if and only if $\mathcal{A}^{\mathbb{N}}$ is of finite type. In this case, the number β is a *simple Parry number* ([42], [19]). A *Pisot number* $\beta > 1$ is a real algebraic integer whose other conjugates have modulus less than one. A *Salem number* $\beta > 1$ is a real algebraic integer whose other conjugates have modulus not greater than one and also one of the conjugates has modulus exactly one. Denote by $\mathbb{R}_+$ the non negative real numbers.

Theorem 2 (Bertrand [18], Schmidt [50])**.** *If β is a Pisot number then each element of $\mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an eventually periodic beta expansion.*

Proof. We denote by $\beta^{(j)}$ $(j = 1, \dots, d)$ the conjugates of β with $\beta^{(1)} = \beta$ and use the same symbol to express the conjugate map $\mathbb{Q}(\beta) \to \mathbb{Q}(\beta^{(j)})$ which sends $x \mapsto x^{(j)}$. As the conjugate map does not increase the denominator of element $x \in \mathbb{Q}(\beta)$, it is enough to show that $T_\beta^n(x)^{(j)}$ is bounded for all j. (The number of lattice points in the bounded region is finite.) This is trivial for $j = 1$ by definition. For $j > 1$, we have

$$T_\beta^n(x) = \beta^n x - \sum_{i=1}^{n} x_i \beta^{n-i}$$

with $x_i \in \mathcal{A}$. Thus

$$\left| T_\beta^n(x)^{(j)} \right| < |x| + \left| \sum_{i=1}^{n} x_i (\beta^{(j)})^{n-j} \right| < |x| + \frac{\lfloor \beta \rfloor}{1 - |\beta^{(j)}|}$$

since $|\beta^{(j)}| < 1$ for $j > 1$. □

Hence a Pisot number is a Parry number. In [50], a partial converse is shown that if all rational number in $[0, 1)$ has an eventually periodic beta expansion then β is a Pisot or a Salem number. It is not yet known whether each element of $\mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an eventually periodic expansion if β is a Salem number (Boyd [20], [21], [22]). See Figure 1 for a brief summary. The finiteness will be discussed in §4.

A Parry number β is also a real algebraic number greater than one, and other conjugates are less than $\min\{|\beta|, (1 + \sqrt{5})/2\}$ in modulus ([42], Solomyak [53]) but the converse does not hold. It is a difficult question to characterize Parry numbers among algebraic integers. ([25], [15])

Hereafter we simply say *Pisot number system* to call the method to express real numbers by beta expansion in Pisot number base. The results like [50] and [18] suggest that Pisot number system is very close to the usual decimal expansion.

2. Delone set and β-integers

Let X be a subset of $\mathbb{R}^d$. The ball of radius $r > 0$ centered at x is denoted by $B(x, r)$. A point x of X is *isolated* if there is a $\varepsilon > 0$ that $B(x, \varepsilon) \cap X = \{x\}$. The set X is called *discrete* if each point of X is isolated. The set X is *uniformly discrete* if there exists a positive $r > 0$ such that $B(x, r) \cap X$ is empty or $\{x\}$ for any $x \in \mathbb{R}^d$, and X is *relatively dense* if there exists a positive $R > 0$ such that $B(x, R) \cap X \neq \emptyset$ for any $x \in \mathbb{R}^d$. A *Delone set* is the set in $\mathbb{R}^d$ which is uniformly discrete and relatively dense at a time. One can expand any positive real number x by beta expansion:

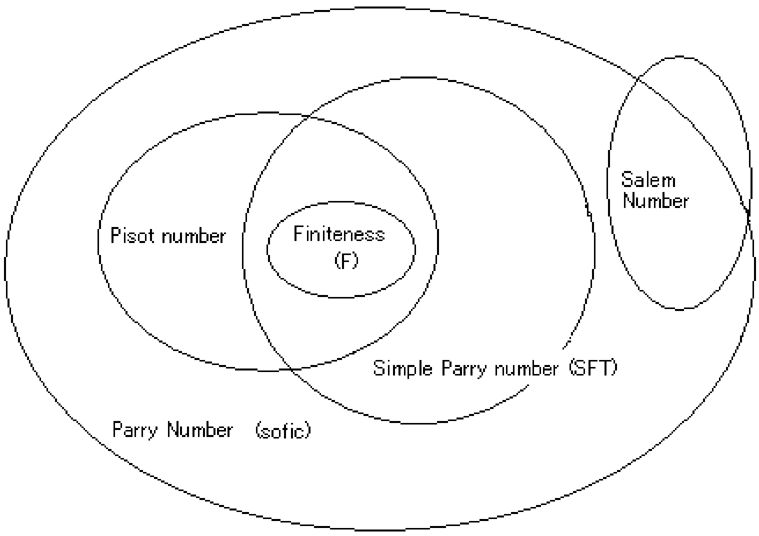

Figure 1. The classification of Parry numbers

$$x = a_{-m}a_{-m+1}\dots a_0 \bullet a_1a_2\dots$$

The β-integer part (resp. β-fractional part) of x is defined by: $[x]_\beta = \pi(a_{-m}\dots a_0)$ (resp. $\langle x\rangle_\beta = \pi(a_1a_2\dots)$). A real number x is a β-*integer* if $\langle|x|\rangle_\beta = 0$. Denote by $\mathbb{Z}_\beta$ the set of β-integers and put $\mathbb{Z}_\beta^+ = \mathbb{Z}_\beta \cap \mathbb{R}_+$.

Proposition 1. *For any $\beta > 1$, the set of β-integers $\mathbb{Z}_\beta$ is relatively dense, discrete and closed in $\mathbb{R}$.*

Proof. As any positive real number x is expressed by beta expansion, one can take $R = 1$ to show that $\mathbb{Z}_\beta^+$ is relatively dense in $\mathbb{R}_+$, which is equivalent to the fact that $\mathbb{Z}_\beta$ is relatively dense in $\mathbb{R}$. Since $\pi(a_{-m}\dots a_0) > \beta^m$ there are only finitely many β-integers in a given ball $B(0,\beta^m)$. Thus $\mathbb{Z}_\beta$ has no accumulation point in $\mathbb{R}$. This proves that $\mathbb{Z}_\beta$ is closed and discrete. $\square$

From now on, we assume that β is not an integer. Then $\lim_{\varepsilon\downarrow 0} T_\beta(1-\varepsilon) = \beta - \lfloor\beta\rfloor \in [0,1)$ and therefore we consider formally [2] the orbit of 1 by the beta transform T_β by putting $T_\beta(1) = \beta - \lfloor\beta\rfloor$. By using (1), it is easily seen that $T_\beta^n(1) = \pi_\beta(\sigma^n(d_\beta(1-0)))$ unless $T_\beta^n(1) = 0$. As $\mathbb{Z}_\beta$ is discrete and closed, we say that $x, y \in \mathbb{Z}_\beta$ is *adjacent* if there are no $z \in \mathbb{Z}_\beta$ between x and y.

Proposition 2. *If $x, y \in \mathbb{Z}_\beta$ is adjacent, then there exists some nonnegative integer n with $|x-y| = T_\beta^n(1)$.*

Proof. To prove this proposition, we use Theorem 1 and transfer the problem into the equivalent one in $\mathcal{A}^{\mathbb{N}}$ under abusive terminology introduced in the previous section. Put $d_\beta(1-0) = c_1c_2\dots$. Without loss of generality, assume that $x > y$, $x = a_{-m}\dots a_0$

[2] 1 is not in the domain of definition of T_β.

with $a_{-m} \neq 0$ and $y = b_{-m} \dots b_0$ by permitting $b_{-m} \dots b_{-m+\ell} = 0^{\ell+1}$. As we are interested in $x-y$, we may assume that $b_{-m} = 0$ since otherwise one can substitute x and y by $(a_{-m} - b_{-m})a_{-m+1} \dots a_0$ and $0b_{-m+1} \dots b_0$. (Both of them are admissible by Theorem 1.) Since x and y are adjacent, $a_{-m} = 1$ since otherwise $(a_{-m} - 1)a_{-m+1} \dots a_0$ is admissible and lies between x and y. Next we see that $a_{-m+1} = 0$ since otherwise $a_m(a_{-m+1} - 1) \dots a_0$ is between x and y. In the same manner, we see that $x = 10^m$. If $b_{-m+1} \dots b_0 <_{\text{lex}} c_1c_2 \dots c_m$ then $c_1c_2 \dots c_m$ lies between x and y, by the lexicographical order. This is not possible. Therefore we must have $d_\beta(x) = 10^m.00\dots$ and $d_\beta(y) = 0c_1c_2 \dots c_m.00\dots$. Thus we have

$$
\begin{aligned}
x - y &= \beta^{m+1}(\pi(10^m) - \pi(0c_1c_2 \dots c_m)) \\
&= \pi(c_{m+1}c_{m+2} \dots) \\
&= \pi(\sigma^m(d_\beta(1-0))) = T^m(1).
\end{aligned}
$$

□

The real number $\beta > 1$ is a *Delone number* [3] if $\{T^n_\beta(1)\}_{n=0,1,2,\dots}$ does not accumulates to 0. If β is a Delone number, then $\mathbb{Z}_\beta$ is uniformly discrete with $r = \min_{n=0,1,\dots} T^n_\beta(1)$. With the help of Proposition 1 and 2, $\mathbb{Z}_\beta$ is a Delone set if and only if β is a Delone number. It is clear that a Pisot number is a Delone number since eventually periodicity of $d_\beta(1-0)$ is equivalent to the fact that $\{T^n_\beta(1)\}_{n=0,1,2,\dots}$ is a finite set. Verger-Gaugry proposed a working-hypothesis that any Perron number is a Delone number (c.f. [19], [57], [29]). However it is not yet known whether there exists an algebraic Delone number which is not a Parry number. By ergodicity, when we fix a β, $d_\beta(x)$ is almost 'normal' with respect to the invariant measure. This means that $\{T^n_\beta(x)\}_{n=0,1,2,\dots}$ is dense in $[0,1)$ for almost all x. Therefore one might also make a completely opposite prediction that an algebraic Delone number is a Parry number. Schmeling [49] had shown a very subtle result that the set of Delone numbers has Hausdorff dimension 1, Lebesgue measure 0 and dense but meager in $[1,\infty)$. Which conjecture is closer to the reality?

3. Definition of Pisot dual tiling

For a point $\xi = (\xi_i)_{i\in\mathbb{Z}} = \dots\xi_{-2}\xi_{-1}\xi_0 \bullet \xi_1\xi_2\dots$ in the subshift $(\mathcal{A}_\mathcal{F}, \sigma)$, let us say the left infinite word $\dots\xi_{-2}\xi_{-1}\xi_0\bullet$ the *integer part* and $\bullet\xi_1\xi_2\dots$ the *fractional part* of ξ. To make the situation clear, here we put the decimal point $\bullet$ on the right/left end to express an integer/fractional part. The symbol $\bullet$ should be neglected if we treat them as a word in $\mathcal{A}^*$. If $\xi_{-i} = 0$ for sufficiently large i, the integer part is expressed by a finite word and if $\xi_i = 0$ for large i then the fractional part is written by a finite word.

For an admissible finite or right infinite word $\omega = \omega_1\omega_2\dots$, denote by S_ω the set of finite integer parts $a_{-m}a_{-m+1}\dots a_0\bullet$ such that the concatenation of $a_{-m}a_{-m+1}\dots a_0\bullet$ and ω is admissible, i.e.,

$$S_\omega = \{a_{-m}a_{-m+1}\dots a_0 \bullet \mid a_{-M}a_{-m+1}\dots a_0 \oplus \omega_1\omega_2\dots \text{ is admissible}\}.$$

[3]Probably we may call it also a Bertrand number. See the description of Prop.4.5 in [19].

This set S_ω is the *predecessors set* of ω. It is shown that the number of distinct predecessor sets is finite if and only the subshift is sofic.

Since the realization map $\pi_\beta : \mathcal{A}^{\mathbb{N}} \to [0,1)$ is continuous, the set of fractional parts is realized as a compact set $[0,1)$. However the set of integer parts is not bounded in $\mathbb{R}$. Thurston embedded this set of integer parts into a compact set in the Euclidean space in the case of Pisot number system ([56]). We explain this idea by the formulation of [2] and [4]. Let β be a Pisot number of degree d and $\beta^{(i)}$ $(i = 1, \dots, r_1)$ be real conjugates, $\beta^{(i)}, \overline{\beta^{(i)}}$ $(i = r_1 + 1, \dots, r_1 + r_2)$ be imaginary conjugates where $\beta^{(1)} = \beta$. Thus $d = r_1 + 2r_2$. Define a map $\Phi : \mathbb{Q}(\beta) \to \mathbb{R}^{d-1}$:

$$\Phi(x) = (x^{(2)}, \dots, x^{(r_1)}, \Re x^{r_1+1}, \Im x^{(r_1+1)}, \dots, \Re x^{(r_1+r_2)}, \Im x^{(r_1+r_2)}).$$

It is shown that $\Phi(\mathbb{Z}[\beta] \cap \mathbb{R}_+)$ is dense in $\mathbb{R}^{d-1}$ ([2]). Since β is a Pisot number, $\Phi(S_\omega)$ is bounded by the Euclidean topology. Take a closure of $\Phi(S_\omega + \omega)$ and call it $\mathcal{T}_\omega$. One can also write

$$\mathcal{T}_\omega = \left\{ \Phi(\omega) + \sum_{i=0}^{\infty} a_{-i} \Phi(\beta^i) \;\middle|\; a_{-m}a_{-m+1} \dots a_0 \oplus \omega_1\omega_2 \dots \text{ is admissible} \right\}.$$

A *Pisot unit* is a Pisot number as well as a unit in the ring of algebraic integers in $\mathbb{Q}(\beta)$. If beta expansions ω are taken over all elements of $\mathbb{Z}[\beta] \cap [0,1)$ (i.e. the fractional parts of $\mathbb{Z}[\beta] \cap \mathbb{R}_+$), we trivially have $\mathbb{R}^{d-1} = \overline{\bigcup_\omega \Phi(S_\omega + \omega)}$. If β is a Pisot unit, the compact sets $\overline{\Phi(S_\omega + \omega)}$ form a locally finite covering of $\mathbb{R}^{d-1}$, we get $\mathbb{R}^{d-1} = \bigcup_\omega \mathcal{T}_\omega$ ([4]). This is a covering of $\mathbb{R}^{d-1}$ by $\mathcal{T}_\omega$. If it is a covering of degree one, the predecessor set of a sofic shift is realized geometrically and give us a tiling of $\mathbb{R}^{d-1}$ by finite number of tiles up to translations. Moreover the congruent tile must be translationally identical and this tiling has self-similarity. Indeed we have

$$\beta^{-1} S_\omega = \bigcup_{a \oplus \omega \,:\, \text{admissible}} \left(\frac{a}{\beta} + S_{a \oplus \omega} \right).$$

The sum on the right is taken over all $a \in \mathcal{A}$ so that $a \oplus \omega$ is admissible. The map $z \to \beta^m z$ from $\mathbb{Q}(\beta)$ to itself is realized as an affine map G_m on $\mathbb{R}^{d-1}$ satisfying the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Q}(\beta) & \xrightarrow{\times \beta^m} & \mathbb{Q}(\beta) \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbb{R}^{d-1} & \xrightarrow[G_m]{} & \mathbb{R}^{d-1} \end{array}$$

The explicit form of G_m is

$$G_m(x_2, x_3, \cdots, x_d) = (x_2, x_3, \cdots, x_d) A_m,$$

where A_m is a $(d-1) \times (d-1)$ matrix:

$$A_m = \begin{pmatrix} (\beta^{(2)})^m & & & & & & 0 \\ & (\beta^{(3)})^m & & & & & \\ & & \ddots & & & & \\ & & & (\beta^{(r_1)})^m & & & \\ & & & & B_1 & & \\ & 0 & & & & \ddots & \\ & & & & & & B_{r_2} \end{pmatrix}$$

with

$$B_j = \begin{pmatrix} \Re((\beta^{(r_1+j)})^{-m}) & \Im((\beta^{(r_1+j)})^{-m}) \\ -\Im((\beta^{(r_1+j)})^{-m}) & \Re((\beta^{(r_1+j)})^{-m}) \end{pmatrix}$$

for $j = 1, \dots, r_2$. G_m is contractive if $m > 0$ and expansive if $m < 0$ by a suitable norm on $\mathbb{R}^{d-1}$. Applying G_{-1}, the tile $\mathcal{T}_\omega$ emerges and is subdivided like

$$G_{-1}(\mathcal{T}_\omega) = \bigcup_{a \oplus \omega} \mathcal{T}_{a \oplus \omega}. \tag{2}$$

Therefore the sofic shift is geometrically realized as a self-affine tiling. In [56], under different notation he wrote,

> It does not quite follow that the K_x determines a tiling of S, for they could in principle have substantial overlap. (skip) However, in many cases of this construction, the shingling are tilings, and the tiles are disks'.

Thurston expected that they should give a tiling in many cases, i.e. the degree is one, and $\mathcal{T}_\omega$ may be homeomorphic to a $d-1$ dimensional disk. The former statement is conjectured positively for all Pisot units but the later has many counter examples.

4. Examples in low degree cases

Let us explain the Pisot dual tiling through concrete examples in degree two and three. It is already non trivial in the quadratic case and generates naturally a special type of sturmian sequences and substitutions. Put $\eta = (1+\sqrt{5})/2$ and let θ be a positive root of $x^3 - x^2 - x - 1$. Then both of them are Pisot units and we see $d_\eta(1-0) = (10)^\infty$ and $d_\theta(1-0) = (110)^\infty$. Thus they are simple Parry numbers. Write $\eta' = (1-\sqrt{5})/2$ and $\theta' \in \mathbb{C} \setminus \mathbb{R}$; one of the complex conjugates of θ.

For understanding, let us begin with the tiling of $\mathbb{R}_+$ by the *direct* embedding of fractional parts. Start with the fundamental tile

$$A = \left\{ \sum_{i=1}^{\infty} a_i \eta^{-i} \;\middle|\; a_i \in \{0,1\}, a_i a_{i+1} = 0 \right\}.$$

This is symbolically written as $A = \{\bullet a_1 a_2 \dots\}$. This is nothing but a realization of the fractional parts of X_η by the convergent power series and by the definition of beta expansion we have $A = [0,1]$. Note that $.0101\dots$ is not admissible but the corresponding beta shift X_η does have such right infinite sequence and hence the right end 1 must be included.

Multiplying η to A behaves as a shift on the symbolic space and it yields a set equation:

$$\eta A = A \cup (1+B), \qquad \eta B = A$$

by classifying the left most symbol, 0 or 1. Here we have $B = \{x \in A \mid a_1 = 0\}$. The reason that B has additional restriction is that the left end symbol 1 must be followed by 0 since 11 is forbidden. This gives $B = [0, 1/\eta]$, and $A = [0,1]$ and $1+B = [1, 1+1/\eta]$ are adjacent. One can omit the translation and write B instead of $1+B$. In fact this makes clearer the situation. The tile A grows to the right to AB by the effect of multiplying η which is a concatenation of two tiles of different length. The tile B grows to A by $\times\eta$. This is nothing but a Fibonacci substitution $A \to AB$, $B \to A$ and the half line $\mathbb{R}_+$ is tiled aperiodically like $ABAABABAABAAB\ldots$ which forms the fixed point of Fibonacci substitution. In general, if β is a Parry number, then the corresponding subshift is sofic and one have an aperiodic tiling of $\mathbb{R}_+$ by finite number of tiles through beta expansion. This construction is well known which we coin it a *direct tiling*.

Now we introduce a *dual tiling* by embedding integer parts. The fundamental dual tile is

$$\mathcal{T} = \mathcal{T}_\lambda = \left\{ \sum_{i=0}^{\infty} x_{-i} {\eta'}^i \;\middle|\; x_{-i} \in \{0,1\}, \quad x_{-i}x_{-i-1} = 0. \right\}$$

This extends beta expansion to the opposite direction and symbolically we may write:

$$\{\ldots x_{-3}x_{-2}x_{-1}x_0\bullet\}$$

However it is not convergent in the usual base η, we use η' instead to have the convergence. The geometric feature is sometimes troublesome but in this case it is easy to see $\mathcal{T} = [-1, \eta]$ an interval. Let us make a *right shift* by dividing by η' to have

$$(\eta')^{-1}\mathcal{T} = \mathcal{T} \cup \mathcal{T}_{.1}$$

The set $\mathcal{T}_{.1}$ is symbolically $\{\ldots x_{-3}x_{-2}x_{-1}.1\}$, i.e., the set of right infinite expansion with a fixed fractional part .1. As 11 is forbidden, $x_{-1} = 0$. Therefore

$$(\eta')^{-1}\mathcal{T} = \mathcal{T} \cup (\eta'\mathcal{T} + {\eta'}^{-1})$$

holds. Put $U = \eta'\mathcal{T} = [-1, 1/\eta]$. As ${\eta'}^{-1} = -\eta$ and

$$\eta'\mathcal{T} + {\eta'}^{-1} = [-1-\eta, 1/\eta - \eta] = [-\eta^2, -1],$$

the interval $\mathcal{T}$ grows to $U\mathcal{T}$ by the right shift. The new tile U is concatenated to the left of T. The situation is explained by an monoid anti homomorphism σ on two letters $\{\mathcal{T}, U\}$ (i.e. it satisfies $\sigma(xy) = \sigma(y)\sigma(x)$) with

$$\sigma(\mathcal{T}) = U\mathcal{T}, \quad \sigma(U) = \mathcal{T}.$$

Iterating σ the tile grows like

$$\begin{array}{rl} U & \mathcal{T} \\ U & \mathcal{T}\mathcal{T} \\ U\mathcal{T}U & \mathcal{T}\mathcal{T} \\ U\mathcal{T}U & \mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T} \\ U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U & \mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T} \\ U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U & \mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T} \\ U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U & \mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}U\mathcal{T}\mathcal{T}U\mathcal{T}\mathcal{T} \end{array}$$

The growing direction is alternating and $\mathcal{T}$ goes to the right and to the left each 2 times. This bi-infinite sturmian sequence satisfies several interesting properties. One of the most illuminating might be the *cut sequence*. Prepare a xy lattice together with all horizontal and perpendicular lines passing through integer points. Draw a line $y = x/\eta$ and put the symbol $\mathcal{T}$ on the intersection of each perpendicular lines and the symbol U on that of each horizontal lines. Let us think that at the origin the line pass through very little above it and put $U\mathcal{T}$. Then we get the cut sequence (See figure 2) which is identical to the above mentioned bi-infinite sturmian sequence. This is one of the general property of sturmian sequence and it is occasionally named after this property. (c.f. [23], see [55] for higher dimensional cases).

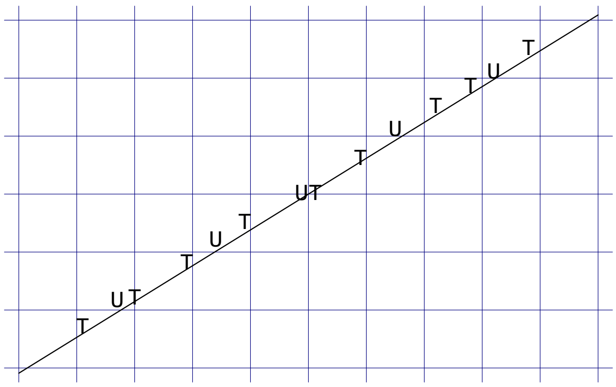

Figure 2. Cut sequence

The essential reason of this phenomenon is that this sequence is a coding of 1-dimensional irrational rotation $x \to \eta' x$. Proceed in the same way in the case of θ. Put

$$\mathcal{T}_\lambda = \left\{ \sum_{i=1}^{\infty} x_{-i}(\theta')^i \;\middle|\; x_{-i} \in \{0,1\}, \quad x_{-i-1} = x_{-i-2} = 1 \Longrightarrow x_{-i-3} = 0 \right\}$$

which is a compact set in the complex plane. Similarly the fundamental tile grows like

$$(\theta')^{-1}\mathcal{T}_\lambda = \mathcal{T}_\lambda \cup \mathcal{T}_{.1}$$
$$(\theta')^{-2}\mathcal{T}_\lambda = \mathcal{T}_\lambda \cup \mathcal{T}_{.1} \cup \mathcal{T}_{.01} \cup \mathcal{T}_{.11}$$
$$(\theta')^{-3}\mathcal{T}_\lambda = \mathcal{T}_\lambda \cup \mathcal{T}_{.1} \cup \mathcal{T}_{.01} \cup \mathcal{T}_{.11} \cup \mathcal{T}_{.001} \cup \mathcal{T}_{.101} \cup \mathcal{T}_{.011}.$$

See the Figure 3.

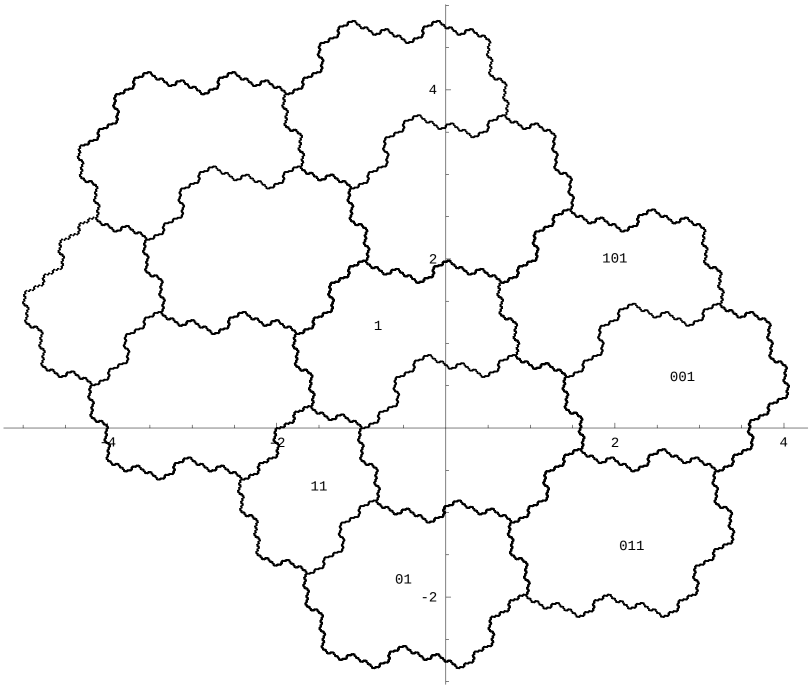

Figure 3. Rauzy Fractal

There are three tiles up to translations. As in the Fibonacci dual case, the origin is an inner point of $\mathcal{T}$, it is shown that the complex plane is aperiodically tiled by these 5 kind of tiles. This tiling may be regarded as a *coding* of the irrational rotation $z \to \theta' z$. Unlike Fibonacci shift, this coding is not realized by words and the geometric nature is not simple ([9]).

Another example by the minimal Pisot number: a root of $x^3 - x - 1$ is shown in Figure 4. In this case, $d_\beta(1-0) = (10000)^\infty$.

5. Finiteness condition implies non overlapping

The property of number systems are intimately related to the tiling introduced in §3, §4. Especially whether they give a tiling, a covering of degree one, or not.

Let $\mathrm{Fin}(\beta)$ be the set of finite beta expansions. $\mathrm{Fin}(\beta)$ clearly consists of non negative element of $\mathbb{Z}[1/\beta]$. (Note that $\mathbb{Z}[\beta] \subset \mathbb{Z}[1/\beta]$ as β is an algebraic integer.) Frougny-Solomyak [27] asked if

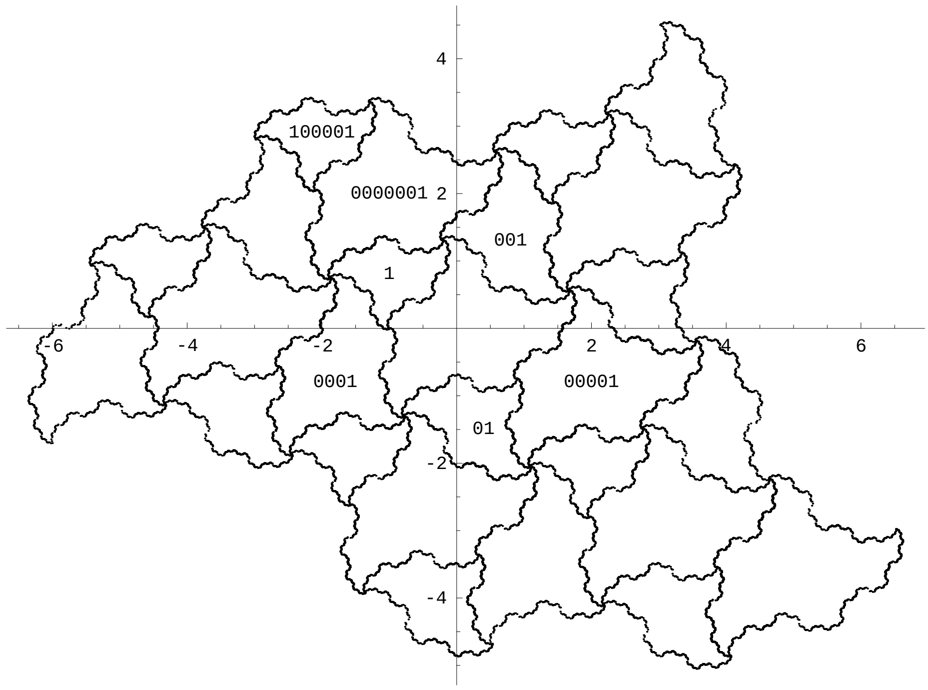

Figure 4. Minimal Pisot case

$$\mathrm{Fin}(\beta) = \mathbb{Z}[1/\beta] \cap \mathbb{R}_+$$

holds or not for a given number system. We say that it satisfies a finiteness condition (F).

A weaker condition $\mathbb{Z} \cap \mathbb{R}_+ \subset \mathrm{Fin}(\beta)$ implies that β is a Pisot number ([3]). Therefore the finiteness (F) holds only when β is a Pisot number. The converse is not true. Especially if the constant term of the minimal polynomial of β is positive, then β has a positive other conjugate and hence (F) does not hold. Further there exists an algorithm to determine whether (F) holds or not ([1]). The relationship is depicted in the Figure 1.

Several sufficient conditions for (F) are also known. For $d_\beta(1) = c_1 c_2 \dots$, if $c_i \geq c_{i+1}$ holds for each i then β is a Pisot number and any number which is expressed as a polynomial of β with non-negative integer coefficients belongs to $\mathrm{Fin}(\beta)$. Additionally if β is a simple Parry number, then β satisfies (F). Let us call this type of β of Frougny-Solomyak type ([27]). Let $x^d - a_{d-1}x^{d-1} - a_{d-2}x^{d-2} - \dots - a_0$ be the minimal polynomial of β and if $a_i \geq 0$ and $a_{d-1} > a_0 + a_1 + \dots + a_{d-2}$ then β satisfies (F). This is called of Hollander type ([33]). The minimal polynomial of cubic Pisot units with the finiteness (F) are classified by the following ([3]):

1. $x^3 - ax^2 - (a+1)x - 1, \ a \geq 0$
2. $x^3 - ax^2 - bx - 1, \ a \geq b \geq 1$ (Frougny-Solomyak type)
3. $x^3 - ax^2 - 1, \ a \geq 1$ (Hollander type)
4. $x^3 - ax^2 + x - 1, \ a \geq 2$

If β is a Pisot unit with the finiteness (F), the origin of $\mathbb{R}^{d-1}$ is an inner point of $\mathcal{T}_\lambda$ and other T_ω $(\omega \neq \lambda)$ does not contain the origin. An inner point of a tile is called *exclusive* if it it does not belong to other tiles. As above, if the origin is an exclusive inner point of $\mathcal{T}$, the tiling is generated by successive use of (2):

$$G_{-n}(\mathcal{T}_\omega) = \bigcup_{a_{-n+1}a_{-n+2}\dots a_0 \oplus \omega} \mathcal{T}_{a_{-n+1}a_{-n+2}\dots a_0 \oplus \omega}.$$

A general Pisot number does not always satisfy this finiteness (F). In such cases, the origin belongs to plural tiles. Even in this case, if the next weaker finiteness is valid, then one can construct the similar tiling:

(W) For any $\varepsilon > 0$ and $z \in \mathbb{Z}[1/\beta] \cap \mathbb{R}_+$ there exist $x, y \in \mathrm{Fin}(\beta)$ such that $z = x - y$ and $|y| < \varepsilon$.

More precisely, let us denote by $\mathcal{P}$ the elements of $\mathbb{Z}[\beta]$ having purely periodic β-expansions. Then the origin is shared by $\mathcal{T}_\omega$ with $\omega \in \mathcal{P}$ and other tiles can not contain 0. Permitting an abuse of terminology, the origin 0 is an exclusive inner point of a union $\bigcup_{\omega \in \mathcal{P}} \mathcal{T}_\omega$. Using this, the condition (W) is equivalent to the fact that the family $\{T_\omega \ : \omega \in [0, 1) \cap \mathbb{Z}[\beta]\}$ forms a covering of $\mathbb{R}^{d-1}$ of degree one, i.e. a tiling. Especially under (W), the boundary of $\mathcal{T}_\omega$ has $(d-1)$-dimensional Lebesgue measure zero ([4]):

Theorem 3 ([2],[4]). *Let β is a Pisot unit with the property (W). Then*

$$\mathbb{R}^{d-1} = \bigcup_{\omega \in \mathbb{Z}[\beta] \cap [0,1)} T_\omega$$

is a tiling.

This weak finiteness (W) is believed to be true for all Pisot numbers (Sidorov [51], [52]), which is an important unsolved problem. In [8], (W) is shown for several families of Pisot numbers, including cubic Pisot units.

For an example of the Pisot unit β with the property (W) but not (F), let β is a cubic Pisot unit defined by $x^3 - 3x^2 + 2x - 1$. It gives a tiling in Figure 5. In this case $d_\beta(1 - 0) = 201^\infty$ and we denote by $\omega = 1^\infty = 111\dots$.

The condition (F) was applied in many different contexts (c.f. [54], [13], [34], [17], [28]). Characterization of Pisot numbers with the property (F) among algebraic integers is an important difficult problem. One can transfer this problem to a problem of the *shift radix system* (SRS for short), a concrete and simple dynamical system on $\mathbb{Z}^{d-1}$. In fact, SRS unifies two completely different number systems: Pisot number systems and *canonical number systems*. The study of SRS is an ongoing project for us ([5], [6], [7], [11], I recommend [12] for the first access).

6. Natural extension and purely periodic orbits

For a given measure theoretical dynamical system $(X, T_1, \mu_1, \mathcal{B}_1)$, if there exists an *invertible* dynamical system $(Y, T_2, \mu_2, \mathcal{B}_2)$ such that $(X, T_1, \mu_1, \mathcal{B}_1)$ is a factor of $(Y, T_2, \mu_2, \mathcal{B}_2)$ then $(Y, T_2, \mu_2, \mathcal{B}_2)$ is called a *natural extension* of $(X, T_1, \mu_1, \mathcal{B}_1)$. There is a general way to construct a natural extension due to Rohlin [46]. However

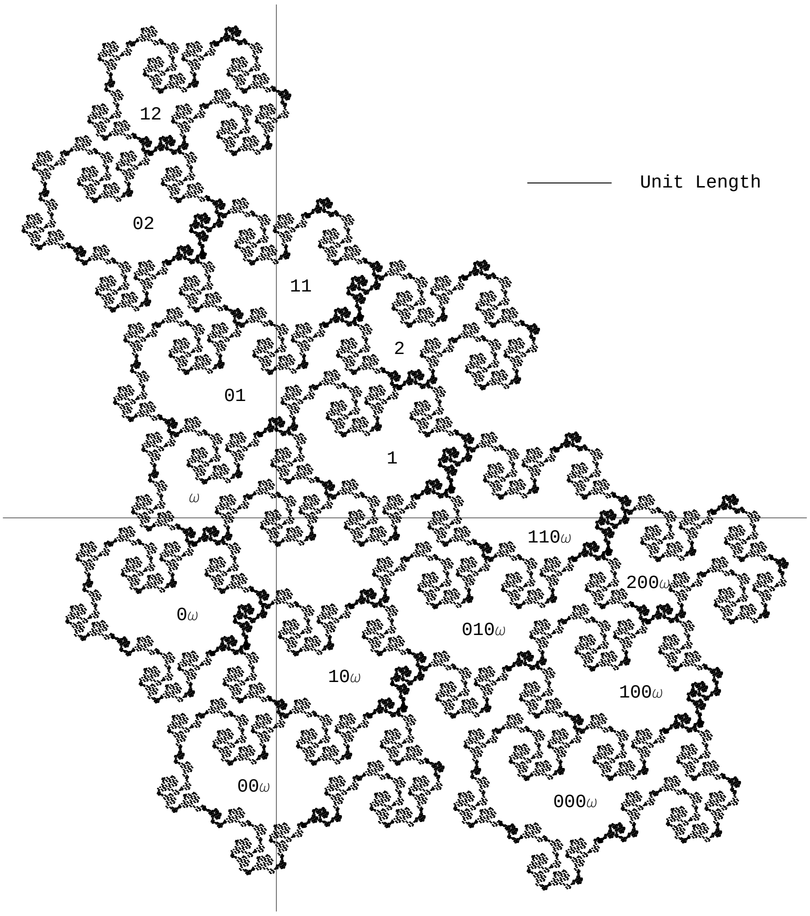

Figure 5. $x^3 - 3x^2 + 2x - 1$

if you wish to answer number theoretical problems, a small and *good* extension is expected, which keeps its algebraic property of the system. Pisot dual tiling gives a way to construct such a natural extension of $([0,1), T_\beta)$ equipped with the Parry measure.

Assume that β is a Pisot unit with the property (W). As β is a Parry number, the set $\{T_\beta^n(1) \mid n = 0, 1, 2, \dots\}$ is finite. Number them like

$$0 < t_1 < t_2 < \cdots < t_\ell = 1$$

and set $t_0 = 0$. Take a $u \in [t_i, t_{i+1})$. Then by Theorem 1 and the construction of Pisot dual tiling, $\mathcal{T}_u - \Phi(u) = \overline{\Phi(S_u)}$ does not depend on the choice of u. Introduce

$$\widehat{X}_\beta = \bigcup_{i=0}^{\ell-1} (-\mathcal{T}_{t_i} + \Phi(t_i)) \times [t_i, t_{i+1})$$

and the map acting on $\widehat{X}_\beta$:

$$\widehat{T}_\beta \;:\; \widehat{X}_\beta \ni (x,y) \to (G_1(x) - \Phi(\lfloor \beta y \rfloor), \beta y - \lfloor \beta y \rfloor) \in \widehat{X}_\beta.$$

and consider the restriction of the Lebesgue measure μ_d on $\mathbb{R}^d$ and the collection $\mathcal{B}$ of Lebesgue measurable sets. Then $\widehat{T}_\beta$ preserves the measure since β is a unit and $(\widehat{X}_\beta, \widehat{T}_\beta, \mu_d, \mathcal{B})$ gives an invertible dynamical system. This extended dynamical system gives a 'bi-infinite' extension of $([0,1), T_\beta)$ and is a factor of the beta shift X_β and the following diagram commutes:

$$\begin{array}{ccc} X_\beta & \xrightarrow{\sigma} & X_\beta \\ \phi\Big\downarrow & & \Big\downarrow\phi \\ \widehat{X}_\beta & \xrightarrow{\widehat{T}_\beta} & \widehat{X}_\beta \\ \mathrm{res}\Big\downarrow & & \Big\downarrow\mathrm{res} \\ [0,1) & \xrightarrow[T_\beta]{} & [0,1) \end{array} \tag{3}$$

where

$$\phi(\dots a_{-1}a_0 \bullet a_1 a_2 \dots) = \left(\lim_{m\to\infty} -\Phi(a_{-m}\dots a_0\bullet),\ \pi_\beta(\bullet a_1 a_2 \dots)\right).$$

and $\mathrm{res}(x,y) = y$.

This extension is realized in the d-dimensional Euclidean space and *good* since Φ is an additive homomorphism defined through conjugate maps, which are ring homomorphisms. By definition $\widehat{X}_\beta$ consists of several cylinder sets $(-\mathcal{T}_{t_i} + \Phi(t_i)) \times [t_i, t_{i+1})$, and this natural partition gives a Markov partition. The Parry measure of $([0,1), T_\beta)$ is retrieved as a restriction of the Lebesgue measure μ_d.

As an application, the purely periodic orbits of T_β is completely described. Using our formulation, we have

Theorem 4 ([32], [31], [38], [37])**.** *An element $x \in \mathbb{Q}(\beta) \cap [0,1)$ has a purely periodic β-expansion if and only if $(\Phi(x), x) \in \widehat{X}_\beta$.*

$\widehat{T}_\beta$ is almost one to one. The main part of the proof of this Theorem is to discuss the intersection of two cylinder sets, the boundary problem. In fact, this is always a problem for a Markov partition. As we wish to have an exact statement, such set of measure zero is not negligible.

In this case, we can show that there are no elements $\mathbb{Q}(\beta) \cap [0,1)$ on such intersection. To show this, the main idea is simple. As $\widehat{X}_\beta$ is compact, there are finite points in $\widehat{X}_\beta$ which correspond to elements of $\mathbb{Q}(\beta) \cap [0,1)$ having a fixed denominator. We can easily show that $\widehat{T}_\beta$ is surjective. But surjectivity and injectivity are equivalent for a finite set. Therefore $\widehat{T}_\beta$ is bijective on the set of points in $\widehat{X}_\beta$ which correspond to $\mathbb{Q}(\beta)\cap[0,1)$. On the other hand, bijectivity breaks down only on the cylinder intersection.

To know more on periodic orbits, we need to give an explicit shape of $\widehat{X}_\beta$. If β is a quadratic Pisot unit, $\widehat{X}_\beta$ is a union of two rectangles and the shape is quite easy. For cubic or higher degree Pisot units, the tile has a fractal boundary. We shall discuss a way to characterize the boundary in the last section. For non unit Pisot numbers, we also have to take into account the p-adic embedding (c.f. [16]).

As the Markov partition based on number systems are simple and concrete, when the topological structure is not complicated, one can deduce geometric information from algebraic consideration on number systems and conversely from the fractal nature of tiles we deduce some number theoretical outcome. For example, in [1] it is shown that

Theorem 5. *If a Pisot unit β satisfies* (F)*, the beta expansion of sufficiently small positive rational numbers is purely periodic.*

This is just a consequence of the fact that the origin is an exclusive inner point of T_λ under (F) condition. For a concrete case, we can show a strange phenomenon ([10]):

Theorem 6. *For a minimal Pisot number θ, the supremum of c that all elements of $[0, c] \cap \mathbb{Q}$ is pure is precisely computed as $0.66666666608644067488\ldots$. Moreover there exists an increasing sequence $a_0 < a_1 < a_2 < \ldots$ lying in $(0, 1)$, that all rationals in $[a_{4i}, a_{4i+1}]$ is not pure and all rational is $[a_{4i+2}, a_{4i+3}]$ is pure.*

The later statement reflects the fractal structure of the boundary of $\mathbb{T}_\lambda$ and perhaps it is not so easy to obtain this conclusion in a purely algebraic manner. This type of tight connection between fractal geometry and number theory is one of the aim of our research.

7. Periodic Tiling and Toral automorphism

Arnoux-Ito [14] realized Pisot type substitutions in a geometric way to higher dimensional irrational rotations. It is also applied to higher dimensional continued fractions. The idea dates back to Rauzy [44], and the fractal sets arises by this construction is widely called *Rauzy fractal* (c.f. [14], [36], [24], [43], [26]). The addition of 1 on the number system is realized as a domain exchange acting on the central tile T_λ of the aperiodic tiling defined in the previous sections. Further, according to their theory, we can tile the space $\mathbb{R}^{d-1}$ periodically (!) as well by the central tile T_λ and its translates under a certain condition. The multiplication by β in the number system gives rise to an explicit construction of Markov partition of automorphisms of $(\mathbb{R}/\mathbb{Z})^d$ associated to the companion matrix of the Pisot unit β. For this construction, the existence of the periodic tiling is essential. Therefore it is worthy to give a direct construction of periodic tiling from the view point of β-expansion. This section is devoted to this task.

Let β be a Pisot unit of degree d with the property (W). A crucial assumption in this section is that cardinality of $\{T_\beta^n(1) \mid n = 0, 1, \ldots\} \setminus \{0\}$ is equal to d. (By considering the degree of the minimal polynomial of β, the cardinality is not less than d.) Set $d_\beta(1 - 0) = c_1 c_2 \ldots$. It is easy to see that $\{1, T_\beta(1), T_\beta^2(1), \ldots, T_\beta^{d-1}(1)\}$ forms a base of $\mathbb{Z}[\beta]$ as a $\mathbb{Z}$-module. Put $r_n = 1 - T_\beta^n(1) = 1 - \pi(c_{n+1} c_{n+2} \ldots)$ and

$$W(\beta) = \left\{ \sum_{i=0}^{d-1} f_i T_\beta^i(1) \;\middle|\; f_i \in \mathbb{Z}, \quad f_0 + f_1 + \cdots + f_{d-1} \geq 0 \right\}.$$

Similarly as $\mathbb{Z}[\beta] \cap \mathbb{R}_+$, one may identify $W(\beta)$ with lattice points in $\mathbb{Z}^d$ lying above a fixed hyperplane and $\Phi(W(\beta))$ is dense in $\mathbb{R}^{d-1}$.

Lemma 1. $P := \{\sum_{i=0}^k b_i \beta^i \mid b_i \in \mathbb{Z}_+\} \subset W(\beta)$

Proof. Consider the regular representation of the multiplication by β with respect to the basis $\{1, T_\beta(1), \dots, T_\beta^{d-1}(1)\}$. As $T_\beta^{j+1}(1) = \beta T_\beta^j(1) - c_{j+1}$, one have

$$\beta \begin{pmatrix} 1 \\ T_\beta(1) \\ T_\beta^2(1) \\ \vdots \\ T_\beta^{d-1}(1) \end{pmatrix} = \begin{pmatrix} c_1 & 1 & & & \\ c_2 & & 1 & & \\ c_3 & & & 1 & \\ \vdots & & & & \ddots \\ c_d & & * & & \end{pmatrix} \begin{pmatrix} 1 \\ T_\beta(1) \\ T_\beta^2(1) \\ \vdots \\ T_\beta^{d-1}(1) \end{pmatrix}$$

where '*' are filled by zeros but 1 appears at most once. The associated matrix is non negative, we are ready. $\square$

$W(\beta) \cap (\mathbb{Z}[\beta] \cap \mathbb{R}_+)$ correspond to lattice points in the cone given by the intersection of two hyperplanes. Lemma 1 supplies a large subset in this intersection. Figure 6 shows the regions in the case of $\beta = (1+\sqrt{5})/2$ where (x, y) corresponds to $x + \beta y$.

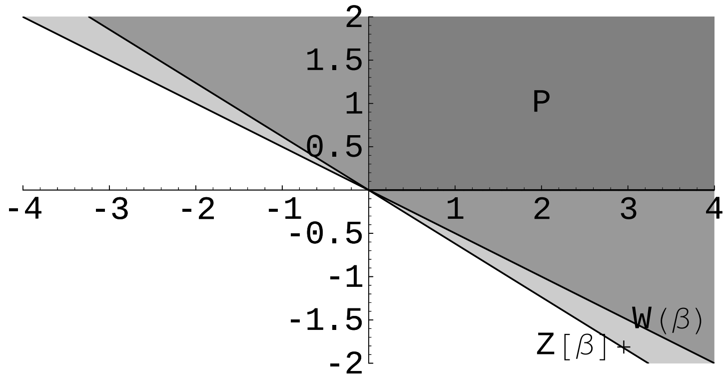

Figure 6. $W(\beta)$ and $\mathbb{Z}[\beta]_+$

Proposition 3. *The set of β-integers forms a complete representative system of $W(\beta)$ $(\mathrm{mod}\ r_1\mathbb{Z} + r_2\mathbb{Z} + \cdots + r_{d-1}\mathbb{Z})$.*

Proof. As shown in Proposition 2, $\mathbb{Z}_\beta^+$ is a uniformly discrete set in $\mathbb{R}_+$ that the distance of adjacent points are in $\{1, T_\beta(1), T_\beta^2(1), \dots, T_\beta^{d-1}(1)\}$. Therefore [4] $\mathbb{Z}_\beta^+ \subset W(\beta)$. We write

$$\mathbb{Z}_\beta^+ = \{z_0, z_1, z_2, \dots \mid z_i < z_{i+1}\}$$

[4] This proves Lemma 1 again.

and consider the order-preserving bijection $\iota : \mathbb{Z}_\beta^+ \to \{0,1,2,\dots\}$ defined by $z_i \mapsto i$. Note that by taking modulo $r_1\mathbb{Z}+r_2\mathbb{Z}+\dots+r_{d-1}\mathbb{Z}$, all the distance of adjacent points are identified with 1. Therefore the image of the map ι is uniquely determined by

$$\iota(z) \equiv z \pmod{r_1\mathbb{Z}+r_2\mathbb{Z}+\dots+r_{d-1}\mathbb{Z}}.$$

On the other hand, for any element $w=\sum_{i=0}^{d-1} f_i T_\beta^i(1) \in W(\beta)$, there exists a unique non negative integer k such that $w \equiv k \pmod{r_1\mathbb{Z}+r_2\mathbb{Z}+\dots+r_{d-1}\mathbb{Z}}$ given by $k=\sum_{i=0}^{d-1} f_i$. □

The next Corollary seems interesting of its own.

Corollary 1. *For any $x \in \mathbb{Z}[\beta]$, there exist a unique $y \in \mathbb{Z}_\beta$ such that $x \equiv y \pmod{r_1\mathbb{Z}+r_2\mathbb{Z}+\dots+r_{d-1}\mathbb{Z}}$.*

Proof. By definition, $\mathbb{Z}[\beta] = -W(\beta)\cup W(\beta)$. Therefore we can naturally extend the map ι in the proof of Proposition 3 to: $\iota : \mathbb{Z}[\beta] \to \mathbb{Z}$ and the assertion follows. □

By this Proposition 3, through the map Φ we have

$$\Phi(W(\beta)) = \bigcup_{(m_1,\dots,m_{d-1})\in\mathbb{Z}^{d-1}} \Phi(\mathbb{Z}_\beta^+) + \sum_{i=1}^{d-1} m_i \Phi(r_i).$$

Taking the closure in $\mathbb{R}^{d-1}$ we get a periodic locally finite covering:

$$\mathbb{R}^{d-1} = T_\lambda + \Phi(r_1)\mathbb{Z} + \dots + \Phi(r_{d-1})\mathbb{Z}.$$

Theorem 7. *If β is a Pisot unit with (W) and the cardinality of $\{T_\beta^n(1) \mid n=0,1,\dots\}\setminus\{0\}$ coincides with the degree d of β, then*

$$\mathbb{R}^{d-1} = T_\lambda + \Phi(r_1)\mathbb{Z} + \dots + \Phi(r_{d-1})\mathbb{Z}.$$

forms a periodic tiling.

Proof. Take an element $w \in W(\beta)\setminus\mathbb{Z}_\beta^+$. We wish to prove that $\Phi(w)$ is not an inner point of T_λ. Assume the contrary that $\Phi(w)$ is an inner point.

First we prove the case when $w>0$. Choose a sufficiently large k such that $\Phi(\beta^k + w) \in \mathrm{Inn}(T_\lambda)$ and $\langle w\rangle_\beta = \langle \beta^k + w\rangle_\beta$. This is always possible. Indeed if the beta expansion of $w>0$ is $a_{-m}\dots a_0 \bullet a_1 a_2 \dots$ with $a_1 a_2 \dots \neq 0^\infty$, then we may choose $k > m+d+1$ such that $\beta^k + w = 10^{k-m-1} \oplus a_{-m}\dots a_0 \bullet a_1 a_2 \dots$ is admissible. This means that $w+\beta^k \notin \mathbb{Z}_\beta^+$. However this is impossible since $\Phi(w+\beta^k) \in T_{a_1 a_2 \dots}$ and we already know that $\{T_\omega : \omega \in \mathbb{Z}[\beta]\cap[0,1)\}$ forms a tiling by Theorem 3. This proves the case $w>0$. Second, assume that $w<0$. Recall that $\beta^k \in W(\beta)$ for $k=0,1,\dots$ by Lemma 1. By Proposition 3 there exists $0 \neq (m_1,\dots,m_{d-1}) \in \mathbb{Z}^{d-1}$ and $y \in \mathbb{Z}_\beta^+$ such that $w = y + \sum m_i r_i$ with the beta expansion $y = a_{-m}\dots a_0\bullet$. Choose k as above, then $\Phi(\beta^k + w)$ is still an inner point of T_λ and $\beta^k + y = 10^{k-m-1} \oplus a_{-m}\dots a_0\bullet$ is admissible. Then $\beta^k + w = \beta^k + y + \sum m_i r_i \notin \mathbb{Z}_\beta^+$ by the uniqueness of the expression of Proposition 3. Therefore without loss of generality we reduce the problem to the first case that $w>0$. □

Coming back to the example θ, the root of x^3-x^2-x-1. Then $r_1 = 1-T_\theta(1) = \theta^{-3}$ and $r_2 = 1 - T_\theta^2(1) = \theta^{-2} + \theta^{-3}$ and we have a periodic tiling:

$$\mathbb{C} = T_\lambda + \theta'^{-3}\mathbb{Z} + (\theta'^{-2} + \theta'^{-3})\mathbb{Z}$$

depicted in Figure 7. In the case x^3-3x^2+2x-1, we have $r_1 = 1-T_\beta(1) = 2\beta^{-1}-\beta^{-2}$ and $r_2 = 1 - T_\beta^2(1) = \beta^{-1} - \beta^{-2}$. Figure 8 is the corresponding figure.

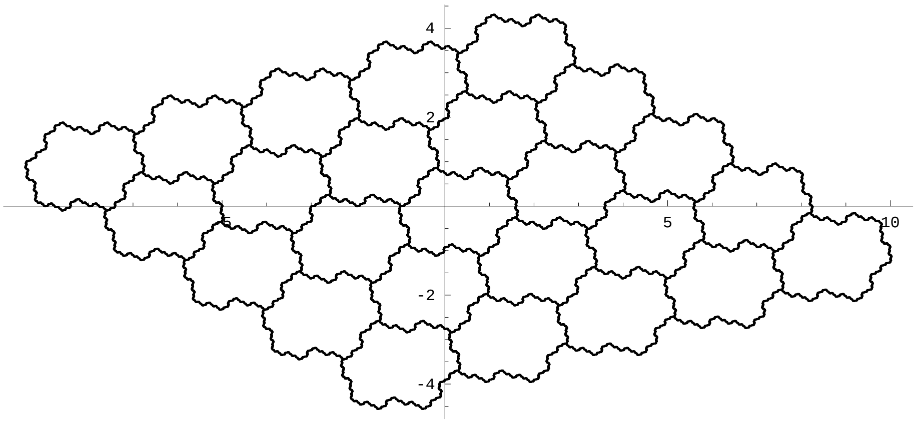

Figure 7. Periodic Rauzy Tiling

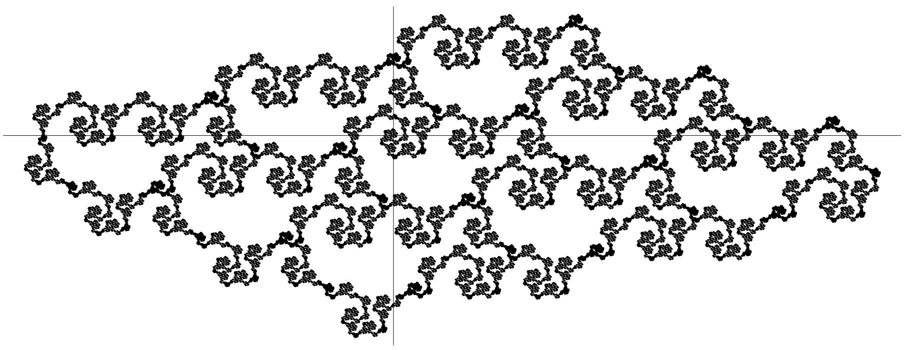

Figure 8. Periodic sofic Tiling

8. Boundary Automaton

The boundary of tiles is captured by a finite state automaton (more precisely a Buchi Automaton which accept infinite words) in several ways. We wish to describe one method, whose essential idea is due to Kátai [35]. Under the condition (W), $\{\mathcal{T}_\omega \;:\; \omega \in \mathbb{Z}[\beta] \cap [0,1)\}$ forms a covering of degree one of $\mathbb{R}^{d-1}$, and the boundary of the tile is a common point of two tiles. Define a labeled directed graph on the vertices $\mathbb{Z}[\beta]$ by drawing edges

$$z_0 \xrightarrow{a|b} z_1$$

whenever two vertices z_0, z_1 satisfy $z_0 = \beta z_1 + a - b$ with $a, b \in \mathcal{A}$. Labels belong to $\mathcal{A} \times \mathcal{A}$. An *essential* subgraph of a directed graph is a subgraph such that each vertex has at least one incoming and also outgoing edge. Take a sufficiently large interval containing the origin and a large constant B. Consider an induced subgraph by vertices that z falls in the interval and $|\Phi(z)| \leq B$. Then the essential graph of this subgraph does not depend on the choice of the interval and B provided they are large enough. Such B and interval is explicitly given:

$$|z| \leq \frac{[\beta]}{\beta - 1} \quad \text{and} \quad |B| \leq \max_{i=2,\dots,d} \frac{[\beta]}{1 - |\beta^{(i)}|}.$$

On the other hand, the admissible infinite word of beta shift is described by an automaton. By a standard technique to make a Cartesian product of two automata, one obtain a finite automaton which recognize common infinite words.

The infinite walks attained in this manner give us the intersection $\mathcal{T}_\lambda \cap \mathcal{T}_\omega$ $(\omega \neq \lambda)$ in terms of infinite words. Therefore it gives the boundary of $\mathcal{T}_\lambda$. By this automaton the boundary of $\mathcal{T}_\omega$ is given as an attractor of a graph directed set. This automaton, called the neighbor automaton, plays an essential role in the study of topological structure of tiles.

If there is a conjugate of β with modulus close to 1, then the size of neighbor automaton becomes huge. This is an obstacle to investigate some property of a family of tiles. If we restrict ourselves to the description of the boundary, there is a better way to make a smaller automaton, the contact automaton (c.f. [30], [47], [48]).

References

[1] S. Akiyama, *Pisot numbers and greedy algorithm*, Number Theory (K. Győry, A. Pethő, and V. Sós, eds.), Walter de Gruyter, 1998, pp. 9–21.

[2] ———, *Self affine tilings and Pisot numeration systems*, Number Theory and its Applications (K. Győry and S. Kanemitsu, eds.), Kluwer, 1999, pp. 1–17.

[3] ———, *Cubic Pisot units with finite beta expansions*, Algebraic Number Theory and Diophantine Analysis (F. Halter-Koch and R. F.Tichy, eds.), de Gruyter, 2000, pp. 11–26.

[4] ———, *On the boundary of self affine tilings generated by Pisot numbers*, J. Math. Soc. Japan **54** (2002), no. 2, 283–308.

[5] S. Akiyama, T. Borbély, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *On a generalization of the radix representation—a survey*, High primes and misdemeanours, Fields Inst. Commun., 2004, Amer. Math. Soc., Providence, R.I., pp. 19–27.

[6] ———, *Generalized radix representations and dynamical systems I*, Acta Math. Hungar. **108** (2005), no. 3, 207–238.

[7] S. Akiyama, H. Brunotte, A. Pethő, and J. M. Thuswaldner, *Generalized radix representations and dynamical systems II*, Acta Arith. **121** (2006), 21–61.

[8] S. Akiyama, H. Rao, and W. Steiner, *A certain finiteness property of Pisot number systems*, J. Number Theory **107** (2004), no. 1, 135–160.

[9] S. Akiyama and T. Sadahiro, *A self-similar tiling generated by the minimal Pisot number*, Acta Math. Info. Univ. Ostraviensis **6** (1998), 9–26.

[10] S. Akiyama and K. Scheicher, *Intersecting two-dimensional fractals with lines*, to appear in Acta Math. Sci. (Szeged).

[11] ______, *Symmetric shift radix systems and finite expansions*, submitted.

[12] ______, *From number systems to shift radix systems*, to appear in Nihonkai Math. J. **16** (2005), no. 2.

[13] P. Ambrož, Ch. Frougny, Z. Masáková, and E. Pelantová, *Arithmetics on number systems with irrational bases*, Bull. Belg. Math. Soc. **10** (2003), 1–19.

[14] P. Arnoux and Sh. Ito, *Pisot substitutions and Rauzy fractals*, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), no. 2, 181–207, Journées Montoises d'Informatique Théorique (Marne-la-Vallée, 2000).

[15] F. Bassino, *Beta-expansions for cubic Pisot numbers*, vol. 2286, Springer, 2002, pp. 141–152.

[16] V. Berthé and A. Siegel, *Purely periodic β-expansions in a Pisot non-unit case*, preprint.

[17] ______, *Tilings associated with beta-numeartion and substitutions*, Elect. J. Comb. Number Th. **5** (2005), no. 3.

[18] A. Bertrand, *Développements en base de Pisot et répartition modulo* 1, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 6, A419–A421.

[19] F. Blanchard, *β-expansions and symbolic dynamics*, Theoret. Comput. Sci. **65** (1989), no. 2, 131–141.

[20] D. W. Boyd, *Salem numbers of degree four have periodic expansions*, Number theory, Walter de Gruyter, 1989, pp. 57–64.

[21] ______, *On beta expansions for Pisot numbers*, Math. Comp. **65** (1996), 841–860.

[22] ______, *On the beta expansion for Salem numbers of degree 6*, Math. Comp. **65** (1996), 861–875.

[23] D. Crisp, W. Moran, A. Pollington, and P. Shiue, *Substitution invariant cutting sequences*, J. Theor. Nombres Bordeaux **5** (1993), no. 1, 123–137.

[24] H. Ei, Sh. Ito, and H. Rao, *Atomic surfaces, tilings and coincidences II: Reducible case.*, to appear Annal. Institut Fourier (Grenoble).

[25] L. Flatto, J. Lagarias, and B. Poonen, *The zeta function of the beta transformation*, Ergod. Th. and Dynam. Sys. **14** (1994), 237–266.

[26] N. Pytheas Fogg, *Substitutions in dynamics, arithmetics and combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer-Verlag, Berlin, 2002, Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[27] Ch. Frougny and B. Solomyak, *Finite beta-expansions*, Ergod. Th. and Dynam. Sys. **12** (1992), 713–723.

[28] C. Fuchs and R. Tijdeman, *Substitutions, abstract number systems and the space filling property*, to appear Annal. Institut Fourier (Grenoble).

[29] J.-P. Gazeau and J.-L. Verger-Gaugry, *Geometric study of the beta-integers for a Perron number and mathematical quasicrystals*, J. Théor. Nombres Bordeaux **16**, no. 1, 125–149.

[30] K. Gröchenig and A. Haas, *Self-similar lattice tilings*, J. Fourier Anal. Appl. **1** (1994), 131–170.

[31] M. Hama and T. Imahashi, *Periodic β-expansions for certain classes of Pisot numbers*, Comment. Math. Univ. St. Paul. **46** (1997), no. 2, 103–116.

[32] Y. Hara and Sh. Ito, *On real quadratic fields and periodic expansions*, 1989, pp. 357–370.

[33] M. Hollander, *Linear numeration systems, finite beta expansions, and discrete spectrum of substitution dynamical systems*, Ph.D. thesis, University of Washington, 1996.

[34] P. Hubert and A. Messaoudi, *Best simultaneous diophantine approximations of Pisot numbers and Rauzy fractals*, to appear in Acta Arithmetica.

[35] K.-H. Indlekofer, I. Kátai, and P. I. Racskó, *Number systems and fractal geometry*, Probability theory and applications (L. Lakatos and I. Kátai, eds.), Math. Appl., vol. 80, Kluwer Acad. Publ., Dordrecht, 1992, pp. 319–334.

[36] Sh. Ito and H. Rao, *Atomic surfaces, tilings and coincidences I: Irreducible case.*, to appear in Israel J.
[37] ———, *Purely periodic β-expansions with Pisot unit base*, Proc. Amer. Math. Soc. **133** (2005), no. 4, 953–964.
[38] Sh. Ito and Y. Sano, *On periodic β-expansions of Pisot numbers and Rauzy fractals*, Osaka J. Math. **38** (2001), no. 2, 349–368.
[39] Sh. Ito and Y. Takahashi, *Markov subshifts and realization of β-expansions*, J. Math. Soc. Japan **26** (1974), 33–55.
[40] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.
[41] M. Lothaire, *Chapter 7, numeration systems*, Algebraic combinatorics on words, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge University Press, 2002.
[42] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.
[43] B. Praggastis, *Numeration systems and Markov partitions from self-similar tilings*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3315–3349.
[44] G. Rauzy, *Nombres algébriques et substitutions*, Bull. Soc. Math. France **110** (1982), no. 2, 147–178.
[45] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.
[46] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Izv. Akad. Nauk SSSR Ser. Math. **25** (1961), 499–530.
[47] K. Scheicher and J. M. Thuswaldner, *Canonical number systems, counting automata and fractals*, Math. Proc. Cambridge Philos. Soc. **133** (2002), no. 1, 163–182.
[48] ———, *Neighbours of self-affine tiles in lattice tilings.*, Fractals in Graz 2001. Analysis, dynamics, geometry, stochastics. Proceedings of the conference, Graz, Austria, June 2001 (Peter (ed.) et al. Grabner, ed.), Birkhäuser, 2002, pp. 241–262.
[49] J. Schmeling, *Symbolic dynamics for β-shifts and self-normal numbers*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 675–694.
[50] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers*, Bull. London Math. Soc. **12** (1980), 269–278.
[51] N. Sidorov, *Bijective and general arithmetic codings for Pisot toral automorphisms*, J. Dynam. Control Systems **7** (2001), no. 4, 447–472.
[52] ———, *Ergodic-theoretic properties of certain Bernoulli convolutions*, Acta Math. Hungar. **101** (2003), no. 2, 345–355.
[53] B. Solomyak, *Conjugates of beta-numbers and the zero-free domain for a class of analytic functions*, Proc. London Math. Soc. **68** (1994), 477–498.
[54] W. Steiner, *Parry expansions of polynomial sequences*, Integers **2** (2002), A14, 28.
[55] J. Tamura, *Certain sequences making a partition of the set of positive integers*, Acta Math. Hungar. **70** (1996), 207–215.
[56] W. Thurston, *Groups, tilings and finite state automata*, AMS Colloquium Lecture Notes, 1989.
[57] J.-L. Verger-Gaugry, *On lacunary Rényi β-expansions of* 1 *with* $\beta > 1$ *a real algebraic number, Perron numbers and a classification problem*, Prepublication de l'Institute Fourier no.648 (2004).