PRIMITIVE MATRICES OVER POLYNOMIAL SEMIRINGS

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ABSTRACT. An extension of the definition of primitivity of a real nonnegative matrix to matrices with univariate polynomial entries is presented. Based on a suitably adapted notion of irreducibility an analogue of the classical characterization of real nonnegative primitive matrices by irreducibility and aperiodicity for matrices with univariate polynomial entries is given. In particular, univariate polynomials with nonnegative coefficients which admit a power with strictly positive coefficients are characterized. Moreover, a primitivity criterion based on almost linear periodic matrices over dioids is presented.

1. INTRODUCTION

In this paper, we are interested in the behavior of powers of matrices over a commutative semiring. Inside this semiring we fix a subset which we think of as the 'nonnegative' elements and which in turn contains another subset which we think of as the 'positive' elements. Given a matrix with only 'nonnegative' entries we ask for conditions which guarantee that a power of this matrix has only 'positive' entries. In this case we talk of a primitive matrix.

We answer the aforementioned question in two different ways. Our first approach is a generalization of the classical PERRON-FROBENIUS Theorem on real matrices with nonnegative coefficients: Here primitivity is characterized by irreducibility and aperiodicity. Our second approach is based on the theory of almost linear periodic sequences over dioids as introduced by GAVALEC [9]. We introduce a new dioid, namely a variant of the standard max-plus algebra whose elements carry additional information: the first coordinate is the 'arrival time', and the second coordinate taking the values 0 or 1 indicates the 'successiveness' of inputs (see Section 6.3 for the definition). Thus the projection on the first coordinate lies in the usual max-plus algebra. The second coordinate is 1 when the supply of inputs is stable in time, otherwise it is 0. This seemingly minor change causes many troubles and it is hard to apply usual linear algebra techniques in max-plus algebra. However, we find that this generalization fits well in describing our primitivity question over polynomial semirings. We are also expecting applications in the theory of invariants of topological Markov chain [12].

Now we describe our results in more detail. In a first step (Section 2 to Section 5) we collect straightforward extensions of usual Perron-Frobenius theory to our settings. Then we turn to the main topic of this paper and address matrices over polynomials rings. Firstly, we reduce the property of primitivity of a given matrix to an easier to handle property of primitivity of the entries of a suitable power of this matrix (Theorem 5.7 and Corollary 5.8). Secondly, we establish a primitivity criterion for polynomial matrices which corresponds to the classical result, but which needs additional requirements on the behavior of the coefficients of the polynomial entries (Theorem 5.17). Apart from the statements on the primitive exponent Section 2 treats a special case of Section 5.

In a second step (Section 6) we provide a primitivity criterion in terms of our new idempotent semiring. To our knowledge this criterion does not have a classical analogue as it exploits properties of almost linear periodic matrices over dioids. Note that our dioid cannot be embedded into a max-plus algebra; consequences of this fact are illustrated by several examples. Regardless of this difficulty, it turns out that some essential results of GAVALEC [9] can be exploited in our settings.

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In particular, our final result relates primitivity to almost periodicity of a sequence of matrices over this new dioid (Theorem 6.25).

In the Appendix we present a formal treatment of the characterization of primitive real matrices in the vein of Pták [21] and Holladay – Varga [14].

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2. An easy generalization of the classical case

In this section we formulate familiar notions in our surroundings. More specifically, we restate a well-known characterization of primitive matrices (see Theorem 2.6) making use of the classical bound on the exponent of primitive matrices.

Throughout this paper we let R be a unital commutative semiring with $1 \neq 0$.

2.1. **The weighted digraph of a matrix over a semiring.** Various notions of irreducibility of matrices will play an important role in our considerations. Let us first recall a classical notion slightly extended to matrices over semirings[1].

Let $A \in R^{r\times r}$. We associate a directed graph to the matrix A. More specifically, the digraph $\mathcal{G}(A)$ is the weighted digraph $([r], E, v)$ with vertex set $[r]$, arc set

$$E = \left\{(i,j) \in [r]^2 \ : \ A_{ij} \neq 0\right\}$$

and weight function $v : E \to R \setminus \{0\}$ with

$$v(i,j) = A_{ij}$$

for all $(i,j) \in E$. Recall that A is called irreducible if the graph $\mathcal{G}(A)$ is strongly connected[2].

2.2. **Primitive matrices over semirings.** The following definition was introduced in [13] for rings: The nonempty subset $\mathcal{P}$ of R is called a preprime of R if $\mathcal{P}$ is additively and multiplicatively closed.

From now on, we assume that R has characteristic 0, and we let $\mathcal{P}$ be a preprime of R and $\mathcal{P}_+$ be a preprime which is contained in $\mathcal{P} \setminus \{0\}$. We are aiming at studying matrices with entries in $\mathcal{P}$ which admit a power all of whose entries belong to $\mathcal{P}_+$.

For the subset S of R we denote by $S^{r\times r}$ the set of $r \times r$ matrices all of whose entries belong to S; here and in the following r is a fixed positive integer.

Definition 2.1. The matrix $A \in \mathcal{P}^{r\times r}$ is called $\mathcal{P}_+$-primitive if there is some $m \in \mathbb{N}_{>0}$ such that $A^m \in \mathcal{P}_+^{r\times r}$. The least such m is called the (primitive) exponent of A w.r.t. $\mathcal{P}_+$ and denoted by $\gamma_{\mathcal{P}_+}(A)$.

We collect some easy observations.

Remark 2.2. Let $A \in \mathcal{P}^{r\times r}$.

(i) As observed by Frobenius [6, Satz XI] A^m is $\mathcal{P}_+$-primitive for every $m \in \mathbb{N}_{>0}$ provided A is $\mathcal{P}_+$-primitive.
(ii) A is $\mathcal{P}_+$-primitive if and only if for some $k \in \mathbb{N}_{>0}$ the matrix A^k is $\mathcal{P}_+$-primitive.
(iii) If A is $\mathcal{P}_+$-primitive then there is some $m \in \mathbb{N}_{>0}$ such that $A^{km} \in \mathcal{P}_+^{r\times r}$ for all $k \in \mathbb{N}_{>0}$. In particular $A \neq 0$.

[1] Schneider [24] defined irreducibility for matrices over integral domains; a similar concept was introduced by Ryser [22].

[2] The graph Γ is called strongly connected if for all vertices i, j of Γ there is a cycle in Γ which contains both i and j. Instead of 'cycle' the notion 'circuit' is frequently used, e.g. in [23].

2.3. **Irreducibility and aperiodicity.** We introduce some kind of relative irreducibility of matrices. In Section 5.2 we shall need a more restrictive notion of irreducibility suitable for univariate polynomial semirings.

Let $A \in \mathcal{P}^{r\times r}$.

Definition 2.3. A is called $\mathcal{P}_+$-irreducible if for every[3] $i, j \in [r]$ there is some $m \in \mathbb{N}$ such that $(A^m)_{ij} \in \mathcal{P}_+$.

Lemma 2.4. *Let $A \in \mathcal{P}^{r\times r}$ be $\mathcal{P}_+$-irreducible and assume $A \neq 0$.*

(i) *A is irreducible.*
(ii) *For every $n \in \mathbb{N}_{>0}$ and $i \in [r]$ there is some $j \in [r]$ with $(A^n)_{ij} \neq 0$.*
(iii) *For every $n \in \mathbb{N}_{>0}$ and $j \in [r]$ there is some $i \in [r]$ with $(A^n)_{ij} \neq 0$.*

Proof. This can easily be checked. □

Definition 2.5. A is called $\mathcal{P}_+$-aperiodic if the greatest common divisor of the set

$$\{\operatorname{per}_1(A, \mathcal{P}_+), \ldots, \operatorname{per}_r(A, \mathcal{P}_+)\}$$

equals 1 where we denote by $\operatorname{per}_i(A, \mathcal{P}_+)$ the greatest common divisor of the set

$$\{n \in \mathbb{N}_{>0} \;:\; (A^n)_{ii} \in \mathcal{P}_+\}$$

if this set is non-void, and $\operatorname{per}_i(A, \mathcal{P}_+) = \infty$, otherwise[4].

Theorem 2.6. *Let R be a unital commutative semiring of characteristic 0, $\mathcal{P}_+$ a preprime of R, $0 \notin \mathcal{P}_+$ and $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$. Let $r \in \mathbb{N}_{>0}$ and $A \in \mathcal{P}^{r\times r}$.*

(i) *A is $\mathcal{P}_+$-primitive if and only if A is $\mathcal{P}_+$-irreducible and $\mathcal{P}_+$-aperiodic.*
(ii) *If A is $\mathcal{P}_+$-primitive then $\gamma_{\mathcal{P}_+}(A) \leq (r-1)^2+1$, and this bound for the primitive exponent w.r.t. $\mathcal{P}_+$ is optimal.*

Proof. (i) The proof given by Lind – Marcus [16, §4.5] can easily be carried over to our case. Alternatively, put

$$E_{ij} = \{m \in \mathbb{N} \;:\; (A^m)_{ij} \in \mathcal{P}_+\} \qquad (i, j \in [r])$$

and apply Theorem 7.4 (see Appendix).
(ii) The arguments given in [14] or [21] can easily be applied to this purely combinatorial statement. For optimality of the given constant see Wielandt's example [27] (for a proof see e.g. [17]). □

As we are interested in preprimes of univariate polynomial semirings which are induced by preprimes of the coefficient semiring we need the following Proposition 2.7. For the subset S of R we denote by $S[X]$ is the set of univariate polynomials all of whose coefficients belong to S. For instance, $\mathcal{P}_+[X]$ is the set of univariate polynomials all of whose coefficients belong to the preprime $\mathcal{P}_+$ of R, and we think of $\mathcal{P}_+[X]$ as the set of 'positive' polynomials.

Proposition 2.7. *$\mathcal{P}_+[X]$ is a preprime of $R[X]$ contained in the preprime $\mathcal{P}[X]$ of $R[X]$.*

Proof. Obvious. □

Note that $\mathcal{P}_+[X]$ is strictly contained in $\mathcal{P}[X]\setminus\{0\}$; thus Theorem 2.6 cannot directly be carried over. Therefore we start with some preparations for the proof of Theorem 5.17.

For the remainder of this paper we assume

$$\mathcal{P}_+ = \mathcal{P} \setminus \{0\}\,.$$

Note that this implies that $\mathcal{P}$ does not contain zero divisors.

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[3] For $n \in \mathbb{N}_{>0}$ we use the abbreviation $[n] = \{1, \ldots, n\}$.

[4] We adopt the convention that every positive integer divides ∞. Furthermore, $\max\emptyset = -\infty$, $\deg(0) = -\infty$, $(-\infty) \pm (-\infty) = -\infty$, $x + (-\infty) = -\infty$, $-\infty < x$, and $|-\infty| > x$ for all $x \in \mathbb{R}$.

3. Combinatorial equivalence

In this section we introduce an equivalence relation on the set of matrices with entries in $\mathcal{P}[X]$; we think of these matrices as 'nonnegative'. This equivalence relation may help to find an analogue of the cyclic structure of real nonnegative matrices (e.g., see Lind – Marcus [16, §4.5] and Schneider [24] for historical remarks on the discussion of cyclic structures in the theory of matrices).

For $f \in R[X] \setminus \{0\}$ and $k = 0, \dots,$ we denote by $\kappa_k(f)$ the k-th coefficient of f, i.e., $f = \sum_{k=0}^{\deg(f)} \kappa_k(f) X^k \in R[X]$ with the usual convention $\kappa_0(f)X^0 = \kappa_0(f)$. If $k > \deg(f)$ or $f = 0$ we put $\kappa_k(f) = 0$. Thus κ_k is an R-linear function from $R[X]$ to R.

In order to decide whether or not a certain power X^t occurs in a result of a polynomial operation we may simplify this operation in the following way: First we replace all occurring nonzero coefficients by 1 (or any other fixed element of $\mathcal{P}_+$), then we perform the operation and finally we check whether the coefficient of X^t has a nonzero coefficient. To make this argument precise we extend an equivalence relation introduced by Gregory – Kirkland – Pullman [10].

Definition 3.1. The matrices $A, B \in \mathcal{P}[X]^{r\times r}$ are said to be $\mathcal{P}_+$-combinatorially equivalent if for all $i, j \in [r]$ and $k \in \mathbb{N}$ we have[5]

$$\kappa_k(A_{ij}) \in \mathcal{P}_+ \iff \kappa_k(B_{ij}) \in \mathcal{P}_+.$$

In this case we write $A \sim B$.

For later use we introduce the polynomials

$$p_n = 1 + X + \cdots + X^n \qquad (n \in \mathbb{N}).$$

Lemma 3.2. (i) $f \cdot p_n + p_m \sim p_{\max\{m, n+\deg(f)\}} \qquad (f \in \mathcal{P}[X],\ 0 \le \deg(f) \le m+1)$

(ii) $p_n \cdot p_m \sim p_{n+m}$

(iii) *Let* $a \in \mathcal{P}_+$. *Then*

$$(a + aX^k) \cdot p_n \sim a p_{k+n} \qquad (0 \le k \le n+1).$$

Proof. Trivial. □

4. Powers of polynomials over commutative semirings

In this section we treat the simple case $r = 1$, i.e., we deal with powers of polynomials. For convenience, we define $\mathrm{lc}(f) = 0$ if $f = 0$, and $\mathrm{lc}(f)$ the leading coefficient of f, otherwise. Thus we have $\mathrm{lc}(f) \neq 0$ for $f \neq 0$. Further, we put $\mathrm{slc}(f) = 0$ if $\deg(f) < 1$, and $\mathrm{slc}(f) = \kappa_{\deg(f)-1}(f)$ if $\deg(f) \ge 1$. We collect some obvious consequences of this notation.

Lemma 4.1. *Let* $f, g \in \mathcal{P}[X]$.

(i) $\kappa_0(fg) = \kappa_0(f)\kappa_0(g)$.
(ii) $\kappa_1(fg) = \kappa_1(f)\kappa_0(g) + \kappa_0(f)\kappa_1(g)$
(iii) $\mathrm{lc}(fg) = \mathrm{lc}(f)\,\mathrm{lc}(g)$
(iv) $\mathrm{slc}(fg) = \mathrm{slc}(f)\,\mathrm{lc}(g) \ + \ \mathrm{lc}(f)\,\mathrm{slc}(g)$
(v) *Let* $f \in \mathcal{P}[X]$, $\deg(f) > 0$ *and* $m \in \mathbb{N}_{>0}$. *If* $\kappa_1(f) = 0$ *then* $\kappa_1(f^m) = 0$, *and if* $\mathrm{slc}(f) = 0$ *then* $\mathrm{slc}(f^m) = 0$.
(vi) *Let* $f_1, \dots, f_r \in \mathcal{P}[X]$. *Then*

$$\deg(f_1 + \cdots + f_r) = \max\{\deg(f_1), \dots, \deg(f_r)\}.$$

(vii) *Let* $f_1, \dots, f_r \in \mathcal{P}[X] \setminus \{0\}$. *Then*

$$\deg(f_1 \cdots f_r) = \deg(f_1) + \cdots + \deg(f_r).$$

(viii) *Let* $\kappa_0(f) \in \mathcal{P}_+$, $g \in \mathcal{P}_+[X]$ *and* $\deg(f) \le \deg(g) + 1$. *Then* $fg \in \mathcal{P}_+[X]$.
(ix) *Let* $f \in \mathcal{P}[X]$, $g \in \mathcal{P}_+[X]$ *and* $\deg(f) \le \deg(g) + 1$. *Then* $f + g \in \mathcal{P}_+[X]$.

[5] For convenience we sometimes write $A > 0$ if all coefficients of all entries of A belong to $\mathcal{P}_+$.

(x) *Let* $d \in \mathbb{N}$, $p_1, \dots, p_r \in \mathcal{P}_+[X]$ *and* $f_1, \dots, f_r \in \mathcal{P}[X]$ *with* $f_k(0) \in \mathcal{P}_+$ *for some* $k \in \{1, \dots, r\}$*. If*

$$\deg(f_i) \leq 1 + d, \qquad \deg(p_i) \geq d \qquad \textit{for all } i = 1, \dots, r$$

then

$$f_1 p_1 + \cdots + f_r p_r \in \mathcal{P}_+[X].$$

(xi) *Let* $f_1, \dots, f_r \in \mathcal{P}[X], f = f_1 + \cdots + f_n$ *and* $d = \deg(f) > 0$*. Then* $\mathrm{slc}(f) \in \mathcal{P}_+$ *if there is some* $i \in [n]$ *with* $\deg(f_i) = d - 1$ *or* $\deg(f_i) = d$ *and* $\mathrm{slc}(f_i) \in \mathcal{P}_+$*.*

The next lemma plays a crucial role in the proof of Theorem 4.3.

Lemma 4.2. *Let* $n \in \mathbb{N}_{>0}$*.*

(i) *If* $a, b, c, d \in \mathcal{P}_+$ *then*

$$(aX^{n+1} + bX^n + cX + d)^{n-1} \in \mathcal{P}_+[X].$$

(ii) *If* $n \geq 3$ *and* $0 < m < n - 1$ *then*

$$(X^{n+1} + X^n + X + 1)^m \notin \mathcal{P}_+[X].$$

Proof. For $m \in \mathbb{N}_{>0}$ we have the relations

$$(X^{n+1} + X^n + X + 1)^m = ((X^n + 1)(X + 1))^m \sim \Big(\sum_{i=0}^{m} X^{ni}\Big) \cdot p_m \tag{4.1}$$

(i) By the above we may assume $a = b = c = d$. By (4.1) we have

$$(X^{n+1} + X^n + X + 1)^{n-1} \sim \Big(\sum_{i=0}^{n-1} X^{ni}\Big) \cdot p_{n-1} \sim p_{n^2-1}.$$

(ii) The left hand side of (4.1) has degree $(n+1)m$, while the right hand side has at most $(m+1)^2$ nonzero coefficients. As $(n+1)m > (m+1)^2$ the polynomial cannot belong to $\mathcal{P}_+[X]$. □

We are now in a position to characterize polynomials with 'nonnegative' coefficients which admit a power with only 'positive' coefficients.

Theorem 4.3. *Let* R *be a unital commutative semiring of characteristic* 0*,* $\mathcal{P}_+$ *a preprime of* R*,* $0 \notin \mathcal{P}_+$ *and* $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$*. Further, let* $f \in \mathcal{P}[X]$ *and* $d = \deg(f) > 0$*. Then* f *is* $\mathcal{P}_+[X]$*-primitive if and only if* $\kappa_0(f), \kappa_1(f), \kappa_{d-1}(f) \in \mathcal{P}_+$*.*

Proof. If f is $\mathcal{P}_+[X]$-primitive then $\kappa_0(f), \kappa_1(f), \kappa_{d-1}(f) \neq 0$ by Lemma 4.1 and Lemma 4.1 (v). An application of Lemma 4.2 concludes the proof. □

Corollary 4.4. *Let* $f \in \mathcal{P}[X]$ *be* $\mathcal{P}_+[X]$*-primitive.*

(i) $f^n \in \mathcal{P}_+[X]$ *for all* $n \geq \gamma_{\mathcal{P}_+[X]}(f)$*.*

(ii) *If* $\deg(f) \geq 3$ *then*

$$\gamma_{\mathcal{P}_+[X]}(f) \leq \deg(f) - 2,$$

and this bound is optimal.

Proof. (i) is clear by Lemma 4.1 (viii).
(ii) The bound for $\gamma_{\mathcal{P}_+[X]}(f)$ and its optimality are clear by Lemma 4.2. □

Remark 4.5. Theorem 4.3 does not hold if $\mathcal{P}_+ \neq \mathcal{P} \setminus \{0\}$. For instance, if $R = \mathbb{Z}$, $\mathcal{P} = \mathbb{N}$ and $\mathcal{P}_+ = 4\mathbb{N}_{>0}$ then the polynomial $2X + 2$ is $\mathcal{P}_+[X]$-primitive.

Trivially, the primitivity of the sum $f_1 + \cdots + f_n$ does not imply that f_i is primitive for some $i \in [n]$ (e.g., let $f_1 = 1 + X^2$, $f_2 = X$). However, primitive polynomials seem to be easier to handle than 'positive' polynomials: Theorem 4.3 yields the following useful statements on primitive and 'positive' polynomials whose proofs are left to the reader.

Proposition 4.6. (i) *The set of* $\mathcal{P}_+[X]$*-primitive polynomials is closed under addition and multiplication.*

(ii) *Let $f, p \in \mathcal{P}[X]$ and $\deg(f) \leq \deg(p) + 1$. If p is $\mathcal{P}_+[X]$-primitive then $f + p$ is $\mathcal{P}_+[X]$-primitive.*

(iii) *Let $f, p \in \mathcal{P}[X]$ and assume that p is $\mathcal{P}_+[X]$-primitive and non-constant. If $\kappa_0(f) \in \mathcal{P}_+$ then $f \cdot p$ is $\mathcal{P}_+[X]$-primitive. If $\mathrm{slc}(f) \in \mathcal{P}_+$ then $f + p$ is $\mathcal{P}_+[X]$-primitive.*

(iv) *Let $f_1, \dots, f_n \in \mathcal{P}[X]$ and assume that f_1 and*

$$f_1 + f_i \qquad (i = 2, \dots, n)$$

are $\mathcal{P}_+[X]$-primitive. Then $f_1 + \cdots + f_n$ is $\mathcal{P}_+[X]$-primitive.

(v) *Let f be $\mathcal{P}_+[X]$-primitive and $p \in \mathcal{P}_+[X]$ with $\deg(p) \geq \deg(f) - 3$. Then we have $pf \in \mathcal{P}_+[X]$.*

We end this section by an observation which seems to of interest for its own sake. Furthermore, it will be applied in the proof of Theorem 5.7 below.

Theorem 4.7. *Let R be a unital commutative semiring of characteristic 0, $\mathcal{P}_+$ a preprime of R, $0 \notin \mathcal{P}_+$ and $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$. Let $f_i \in \mathcal{P}[X]$ $(i = 1, 2, \dots)$ be non-constant $\mathcal{P}_+[X]$-primitive polynomials such that $\deg(f_i) \leq B$ for a given constant $B \in \mathbb{N}_{\geq 3}$. Then $\prod_{i=1}^{B-2} f_i$ is in $\mathcal{P}_+[X]$.*

Proof. If $f_i = 1 + X$ for all i, then $(1 + X)^{B-2} \sim p_{B-2}$. Let us inductively substitute $f_i = 1 + X$ by $1 + X + X^{m_i - 1} + X^{m_i}$ for $2 \leq m_i \leq B$. First

$$(1 + X + X^{m_1-1} + X^{m_1})(1 + X)^{B-3} = (1 + X)^{B-2} + X^{m_1-1}(1 + X)^{B-2} \sim p_{B+m_1-3}$$

by Lemma 3.2, and second

$$(1+X+X^{m_1-1}+X^{m_1})(1+X+X^{m_2-1}+X^{m_2})(1+X)^{B-4} \sim p_{B+m_1-3}+X^{m_2}p_{B+m_1-3} \sim p_{B+m_1+m_2-3}.$$

Iterating this process, we find

$$\prod_{i=1}^{B-2} (1 + X + X^{m_i-1} + X^{m_i}) \sim p_{B-3+\sum_i m_i}.$$

□

5. Powers of polynomial matrices over commutative semirings

In this section we concentrate on matrices whose entries are univariate polynomials over semirings.

5.1. **Primitivity of polynomial matrices.** For $A \in R[X]^{r \times r}$ and $k \in \mathbb{N}$ we denote by $\kappa_k(A)$ the matrix in $R^{r \times r}$ whose entries are $\kappa_k(A_{ij}) \quad (i, j \in [r])$. Further we set (see [7])

$$\deg(A) = \max\{\deg A_{ij} \ : \ i, j \in [r]\},$$

thus we can write

$$A = A_0 + A_1 X + \cdots + A_d X^d \tag{5.1}$$

with $d = \deg(A)$ and uniquely determined matrices $A_0, \dots, A_d \in R^{r \times r}$ (see [7], [18]). For convenience we set

$$\mathrm{lc}(A) = A_d,$$

and

$$\mathrm{slc}(A) = \begin{cases} A_{d-1}, & \text{if } d > 0, \\ 0, & \text{otherwise.} \end{cases}$$

The statements of the next lemma can easily be verified.

Lemma 5.1. *Let $k \in \mathbb{N}_{>0}$.*

(i)
$$\kappa_0(A^n) = \kappa_0(A)^n.$$

(ii)
$$\kappa_1(A^n) = \sum_{i=0}^{n-1} \kappa_0(A)^{n-1-i}\, \kappa_1(A)\, \kappa_0(A)^i.$$

(iii) *If* $\mathrm{lc}\,(A)$ *is not nilpotent then*

$$\mathrm{lc}\,(A^n) = \mathrm{lc}\,(A)^n$$

and

$$\mathrm{slc}\,(A^n) = \sum_{i=0}^{n-1} \mathrm{lc}\,(A)^{n-1-i}\,\mathrm{slc}\,(A)\,\mathrm{lc}\,(A)^i.$$

Theorem 5.2. *Let R be a commutative unital semiring of characteristic 0, $\mathcal{P}_+$ a preprime of R, $0 \notin \mathcal{P}_+$ and $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$, $r \in \mathbb{N}_{>0}$ and $A \in \mathcal{P}[X]^{r\times r}$. If A is $\mathcal{P}_+[X]$-primitive then the matrix $\big(\kappa_0(A_{ij})\big)_{i,j\in[r]} \in \mathcal{P}^{r\times r}$ is $\mathcal{P}_+$-primitive, and we have*

$$\gamma_{\mathcal{P}_+[X]}(A) \geq \gamma_{\mathcal{P}_+}(\kappa_0(A)).$$

Proof. With the notation of (5.1) we obviously have $A_0 = \kappa_0(A)$, and the result follows. □

Corollary 5.3. *If A is $\mathcal{P}_+[X]$-primitive then for all $i \in [r]$ there is some $j \in [r]$ such that $\kappa_0(A_{ij}) \in \mathcal{P}_+$ (analogously for columns).*

Remark 5.4. (i) For the $\mathbb{R}_{>0}[X]$-primitive matrix

$$A = \begin{pmatrix} 0 & 1+X \\ 1 & 1 \end{pmatrix} \in \mathbb{R}_{\geq 0}[X]^{2\times 2}$$

we have

$$\mathrm{lc}(A) = \kappa_1(A) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathrm{slc}(A) = \kappa_0(A) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note that $\mathrm{lc}\,(A)$ is nilpotent.

(ii) In case $\deg\,(A) > 0$ we cannot replace $\kappa_0(A_{ij})$ by $\kappa_1(A_{ij})$ or $\mathrm{slc}(A_{ij})$ in Corollary 5.3 (e.g., see the matrix A in (i)).

Our next Theorem 5.7 plays a key role in our arguments. It shows that under seemingly strong conditions primitivity of a matrix A can be checked without calculating powers of A. First we derive a lower bound for the degrees of the entries of powers of A. Our argument requires some technical preparations.

For n, $r \in \mathbb{N}_{>0}$ we denote by $\mathcal{M}_n$ the set of finite sums of monomials[6] of the form

$$X_{i_1j_1}^{m_{i_1j_1}} \cdots X_{i_nj_n}^{m_{i_nj_n}} \qquad (i_1,\dots,i_n,j_1,\dots,j_n \in [r],\ m_{i_1j_1},\dots,m_{i_nj_n} \in \mathbb{N}_{>0})$$

where $X_{1,1}, X_{1,2}, \dots, X_{r,r-1}, X_{r,r}$ are r^2 commuting variables. For $p \in \mathcal{M}_n$ and $\nu, \mu \in [r]$ we denote by $\deg_{\nu\mu} p$ the degree of p written as a polynomial in $X_{\nu\mu}$.

The following Lemma is certainly well known, however, we could not find a suitable reference.

Lemma 5.5. *Define recursively a sequence of polynomials by*

$$p_{ij}^{(1)} = X_{ij} \in \mathcal{M}_1$$

and

$$p_{ij}^{(n+1)} = \sum_{k\in[r]} p_{ik}^{(n)} X_{kj} \in \mathcal{M}_{n+1} \qquad (n \geq 1, \quad i,j \in [r]).$$

Then we have for $n \geq 1$ and $i, j \in [r]$:

(i) $\deg_{ii} p_{ii}^{(n)} = n$.

(ii) *If $i \neq j$ then*

$$\deg_{ii} p_{ij}^{(n)} = \deg_{jj} p_{ij}^{(n)} = n - 1.$$

[6] Schneider [24] introduced polynomials which are linear in $X_1, \dots, X_n$ aiming at characterizing reducibility by means of determinants.

(iii) *Let $i \neq j$. If n is even then*

$$\deg_{ij} p_{ii}^{(n)} = \deg_{ij} p_{ij}^{(n)} = \frac{n}{2},$$

and if n is odd then

$$\deg_{ij} p_{ii}^{(n)} = \frac{n-1}{2} \qquad \text{and} \qquad \deg_{ij} p_{ij}^{(n)} = \frac{n+1}{2}.$$

(iv) *For $\nu \in [r] \setminus \{i,j\}$ we have*

$$\deg_{\nu j} p_{ij}^{(n)} = \deg_{i\nu} p_{ij}^{(n)} = \begin{cases} \frac{n}{2} & (n \text{ even}), \\ \frac{n-1}{2} & (n \text{ odd}). \end{cases}$$

(v) *For $\nu \in [r] \setminus \{i\}$, $\mu \in [r] \setminus \{j\}$ we have*

$$\deg_{\nu\mu} p_{ij}^{(n)} = \begin{cases} \frac{n}{2} - 1 & (n \text{ even}), \\ \frac{n-1}{2} & (n \text{ odd}). \end{cases}$$

(vi)

$$\deg_{\nu\mu} p_{ij}^{(n)} \geq \left\lfloor \frac{n-1}{2} \right\rfloor \qquad (\nu, \mu \in [r]).$$

(vii) $p_{ij}^{(n)}(ta_{11}, \ldots, ta_{rr}) = t^n p_{ij}^{(n)}(a_{11}, \ldots, a_{rr})$ *for all $t, a_{11}, \ldots, a_{rr} \in R$.*

Proof. (i) – (v) and (vii) by induction, (vi) by the above. □

Let $A \in R^{r\times r}$. For $p \in \mathcal{M}_n$ we write by abuse of notation

$$p(A) = p(A_{1,1}, A_{1,2}, \ldots, A_{r,r-1}, A_{rr}).$$

Corollary 5.6. *Let $A \in R^{r\times r}$, $n \geq 1$ and $i, j \in [r]$. Then $(A^n)_{ij} = p_{ij}^{(n)}(A)$.*

Now we can formulate a sufficient primitivity condition and a bound for the exponent of primitivity.

Theorem 5.7. *Let R be a unital commutative semiring of characteristic 0, $\mathcal{P}_+$ a preprime of R, $0 \notin \mathcal{P}_+$ and $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$. Let $A \in \mathcal{P}[X]^{r\times r}$ and assume that A_{ij} is $\mathcal{P}_+[X]$-primitive for all $i, j \in [r]$. Then A is $\mathcal{P}_+[X]$-primitive. Furthermore, if $\deg(A) \leq 3$ we have $A \in \mathcal{P}_+[X]^{r\times r}$, otherwise we have*

$$\gamma_{\mathcal{P}_+[X]}(A) \leq 2\deg(A) - 3,$$

and this constant is best possible.

Proof. For $d = \deg(A) \leq 3$ we have nothing to prove, therefore we assume $d \geq 4$. Pick $\nu, \mu \in [r]$ with $d = \deg(A_{\nu\mu})$. Fix $i, j \in [r]$ and let $n \geq 2d - 3$. We infer from Lemma 5.5 (vi) that $\deg_{\nu\mu} p_{ij}^{(n)} \geq \lfloor (n-1)/2 \rfloor$. Therefore, written as a polynomial in the variable $X_{\nu\mu}$ the polynomial $p_{ij}^{(n)}$ has a monomial u with at least

$$\left\lfloor \frac{n-1}{2} \right\rfloor \geq d - 2$$

factors of degree d. The product of these factors belongs to $\mathcal{P}_+[X]$ by Theorem 4.7, thus the same is true for u by Lemma 4.1 (viii). Using Lemma 4.1 (ix) and Corollary 5.6 we conclude

$$(A^n)_{ij} = p_{ij}^{(n)}(A) \in \mathcal{P}_+[X].$$

Finally, the bound $2d - 3$ is best possible by inspecting the $(1,2)$-entry of the matrix

$$\begin{pmatrix} 1 & 1 + X + X^{d-1} + X^d \\ 1 & 1 \end{pmatrix}^{2d-4}$$

□

Note that Remark 5.4 shows that the converse of Theorem 5.7 does not hold. However, we can now reduce the property of primitivity of a matrix A to an easier to handle property of primitivity of the entries of a suitable power of A; certainly, this is a tautology in the cases $\deg(A) = 0$ or $r = 1$.

Corollary 5.8. *A is $\mathcal{P}_+[X]$-primitive if and only if there is some $m \in \mathbb{N}_{>0}$ such that for all $i, j \in [r]$ the entry $(A^m)_{ij}$ is $\mathcal{P}_+[X]$-primitive, i.e., either*

$$(A^m)_{ij} \in \mathcal{P}_+$$

or

$$\deg(A^m)_{ij} \geq 1 \quad \textit{and} \quad \kappa_0(A^m)_{ij}, \quad \kappa_1(A^m)_{ij}, \quad \mathrm{slc}(A^m)_{ij} \in \mathcal{P}_+.$$

Proof. Clear by Theorems 4.3 and 5.7 and Remark 2.2. □

We conclude this subsection by a generalization of an example given by Perron [20]. It shows that in favorable cases primitivity can be seen directly.

Example 5.9. Let $1 \in \mathcal{P}, \ , f_1, \ldots, f_r \in \mathcal{P}[X]$ and

$$A = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & f_1 \\ 1 & 0 & \cdots & \cdots & 0 & f_2 \\ 0 & & & & \vdots & \vdots \\ \vdots & & & & 0 & f_{r-1} \\ 0 & \cdots & \cdots & 0 & 1 & f_r \end{pmatrix}.$$

If f_1 and f_r are $\mathcal{P}_+[X]$-primitive and $f_2, \ldots, f_{r-1}$ are either $\mathcal{P}_+[X]$-primitive or zero then A is $\mathcal{P}_+[X]$-primitive, and we have

$$\gamma_{\mathcal{P}_+[X]}(A) \leq \begin{cases} \max\{1, \deg(A) - 2\} & \text{if} \quad r = 1, \\ 2\max\{1, 4\deg(A) - 3\} & \text{if} \quad r = 2, \\ (r+1)\max\{1, 2(r+1)\deg(A) - 3\} & \text{if} \quad r \geq 3, \end{cases}$$

because the essentials of the proof given in [20] can be carried over to our situation: For $n \in \mathbb{N}_{>0}$ and $i \in [r]$ we have

$$(A^{n+1})_{i,j} = (A^n)_{i,j+1} \quad (0 \leq j \leq r-1), \qquad (A^{n+1})_{i,r} = \sum_{k=1}^{r} (A^n)_{i,k} f_k$$

and by induction using Proposition 4.6 we show that $(A^n)_{ij}$ is either primitive or zero ($j \in [r]$). Similarly, we show $(A^n)_{ir}$ primitive for $n \geq 2, \ i \in [r]$. By induction over k we find that $(A^{n+k})_{i,r-k}$ is primitive for $n \geq 2, \ i \in [r]$ and $0 \leq k \leq r-1$, and derive

$$(A^{r+1})_{ij} \quad \text{primitive} \qquad (i, j \in [r]).$$

From Theorem 5.7 we infer that A^{r+1} is primitive, hence A is primitive. Using

$$\deg(A^m) \leq m \deg(A)$$

the bounds for $\gamma_{\mathcal{P}_+[X]}(A)$ can easily be derived from Corollary 4.4 and Theorem 5.7.

5.2. **Strong irreducibility and strong aperiodicity.** In an attempt to find a direct analogue to the classical Perron-Frobenius Theorem we introduce more restrictive notions of irreducibility and aperiodicity for matrices over polynomial semirings. In the first step we concentrate our attention on the lower coefficients (see Definition 5.13 (i)) and in a second step on the higher coefficients (see Definition 5.15) of the polynomials involved.

We start with an observation on the growth of the degree of a matrix once we are sure that none of its entries vanishes.

Lemma 5.10. *Let $A \in \mathcal{P}[X]^{r \times r}$, $m \in \mathbb{N}_{>0}$ such that $(A^m)_{ij} \neq 0$ for all $i, j \in [r]$. For all $t \in \mathbb{N}_{>0}$ and all $i, j \in [r]$ we have*

$$\deg(A^{2(m+1)t})_{ij} \geq t \cdot \deg(A).$$

Proof. We start by showing $\deg (A^{m+1})_{ij} \geq \deg A_{\ell j}$ for all $i, j, \ell \in [r]$. If $A_{\ell j} = 0$ there is nothing to prove. Otherwise setting $B = A^m$ and using Lemma 4.1 we find

$$\deg (A^{m+1})_{ij} = \deg (BA)_{ij} = \max \{\deg (B_{ik} \cdot A_{kj}) \ : \ k \in [r]\} \geq \deg (B_{i\ell} \cdot A_{\ell j}) \geq \deg A_{\ell j}.$$

Analogously we find $\deg (A^{m+1})_{ij} \geq \deg A_{i\ell}$ for all $i, j, \ell \in [r]$. Thus we have established

$$\deg (A^{m+1})_{ij} \geq \max \{\deg A_{i\ell}, \deg A_{\ell j} \ : \ \ell \in [r]\} \geq 0 \qquad (i, j \in [r]).$$

Pick $i', j' \in [r]$ with $\deg A_{i'j'} = \deg (A)$. Then clearly

$$\deg (A^{m+1})_{ij'} \geq \deg (A) \qquad (i \in [r]).$$

Now we prove our assertion by induction on t. By Lemma 4.1 and what we have seen above we settle the case $t = 1$ by

$$\begin{aligned} \deg (A^{2(m+1)})_{ij} &= \max \left\{\deg ((A^{m+1})_{ik} \cdot (A^{m+1})_{kj}) \ : \ k \in [r]\right\} \\ &\geq \deg ((A^{m+1})_{ij'} \cdot (A^{m+1})_{j'j}) \\ &= \deg (A^{m+1})_{ij'} + \deg (A^{m+1})_{j'j} \ \geq \ \deg (A). \end{aligned}$$

Finally we observe that all entries of the first factor of the product

$$A^{2(m+1)(t+1)} = A^{2(m+1)t} \cdot A^{2(m+1)}$$

have degree at least $t \cdot \deg (A)$ and that all entries of the second factor have degree at least $\deg (A)$. □

Lemma 5.11. *Let $A \in \mathcal{P}[X]^{r\times r}$, $i \in [r]$ and $n \in \mathbb{N}_{>0}$ such that for all $j \in [r]$ we have $\kappa_0((A^n)_{ij}) > 0$. Then for all $j \in [r]$ there is some $k \in [r]$ such that $\kappa_0(A_{kj}) > 0$.*

Proof. Using Lemma 4.1 (i) the assumption of the contrary would lead to the contradiction

$$\kappa_0((A^n)_{ij}) \ = \ \sum_{k\in[r]} \kappa_0((A^{n-1})_{ik}) \, \kappa_0(A_{kj}) \ = \ 0.$$

□

Lemma 5.12. *Let $A \in \mathcal{P}[X]^{r\times r}$ and assume that for all $j \in [r]$ there is some $\ell \in [r]$ such that $\kappa_0(A_{\ell j}) > 0$. Let $i \in [r]$ and $n \in \mathbb{N}$ have the property: For all $j \in [r]$ we have*

$$(A^n)_{ij} > 0 \text{ and } \deg (A^n)_{ij} \geq \deg (A).$$

Then for all $m \geq n$ and $j \in [r]$ we have

$$(A^m)_{ij} > 0 \text{ and } \deg (A^m)_{ij} \geq \deg (A).$$

Proof. We proceed by induction on m. Pick $\ell \in [r]$ such that $\kappa_0(A^m)_{\ell j} > 0$, hence $(A^m)_{i\ell} A_{\ell j} > 0$ by Lemma 4.1 (viii). Then Lemma 4.1 (x) yields

$$(A^{m+1})_{ij} = \sum_{k\in[r]} (A^m)_{ik} A_{kj} > 0,$$

and an application of Lemma 4.1 (vi),(vii) concludes the proof. □

Note that in case deg (A) = 0 the following definitions agree with the classical notions.

Definition 5.13. Let $A \in \mathcal{P}[X]^{r\times r}$.

(i) A is called strongly $\mathcal{P}_+[X]$-irreducible if for all $i, j \in [r]$ there is some $n \in \mathbb{N}$ such that the following three properties are satisfied:
  (a) $\kappa_0(A^n)_{ij} \in \mathcal{P}_+$
  (b) $\deg (A^n)_{ij} \geq \min \{1, \deg (\mathrm{A})\}$
  (c) If $\deg (A^n)_{ij} > 0$ then we have $\kappa_1(A^n)_{ij} \in \mathcal{P}_+$.

(ii) Let $i \in [r]$. We denote by $\operatorname{sper}_i(A)$ the greatest common divisor of the set

$$\{n \in \mathbb{N}_{>0} \,:\, \kappa_0(A^n)_{ii} \in \mathcal{P}_+,\ \deg(A^n)_{ij} \geq \min\{1, \deg(\mathrm{A})\}\,,\ \text{and}\ \deg(A^n)_{ij} > 0 \implies \kappa_1(A^n)_{ij} \in \mathcal{P}_+\}$$

if this set is non-void, and $\operatorname{sper}_i(A) = \infty$, otherwise. A is called strongly $\mathcal{P}_+[X]$-aperiodic if $\gcd\,\{\operatorname{sper}_1(A), \ldots, \operatorname{sper}_r(A)\} = 1$.

Clearly, strong irreducibility implies irreducibility. Note that in case $\deg(\mathrm{A}) > 0$ strong irreducibility and strong aperiodicity do not imply primitivity (e.g., $1 + X + X^3 \in \mathbb{R}_{\geq 0}[X]$ is not $\mathbb{R}_{>0}[X]$-primitive).

For the notions used in the following Lemma we refer the reader to the Appendix.

Lemma 5.14. *Let $A \in \mathcal{P}[X]^{r\times r}$. For $i, j \in [r]$ let E_{ij} be the set of all nonnegative integers which satisfy the three properties (i), (ii) and (iii) of Definition i.*

(i) *If $i \neq j$ then $0 \notin E_{ij}$.*
(ii) *For $i, j \in [r]$, $m \in E_{ij}$ and $n \in E_{jj}$ we have $m + n \in E_{ij}$.*
(iii) *For $i, j \in [r]$, $m \in E_{ij}$ and $n \in E_{ji}$ we have $m + n \in E_{ii}$.*
(iv) *$\{E_{ij}\}_{i,j\in[r]}$ is a suitable family of subsets of $\mathbb{N}$.*
(v) *If A is strongly $\mathcal{P}_+[X]$-irreducible we have for all $i, j \in [r]$:*
  (a) *There exists some $m \in \mathbb{N}$ such that $m + n \in E_{ij}$ for all $n \in E_{jj}$.*
  (b) *If $i \neq j$ then there exists $s \in E_{ij}$ and $t \in E_{ji}$ such that $s + n + t \in E_{ii}$ for all $n \in E_{jj} \cup \{0\}$.*
(vi) *If A is strongly $\mathcal{P}_+[X]$-irreducible and strongly $\mathcal{P}_+[X]$-aperiodic then there exists some $m \in \mathbb{N}_{>0}$ such that $n \in E_{ij}$ for all $n \geq m$ and all $i, j \in [r]$.*

Proof. (i) The assumption $0 \in E_{ij}$ yields the contradiction $0 = \kappa_0(0) = \kappa_0(A^0)_{ij} \in \mathcal{P}_+$.
(ii) Write

$$(A^{m+n})_{ij} = (A^m)_{ij}(A^n)_{jj} + h$$

with some $h \in \mathcal{P}[X]$. By Lemma 4.1 we find $\kappa_0((A^{m+n})_{ij}) \in \mathcal{P}_+$. Further, by Lemma 4.1 (vi)

$$\deg(A^{m+n})_{ij} \geq \deg(A^m)_{ij} + \deg(A^n)_{jj} \geq \min\{1, d\}\,.$$

If $\deg(A^{m+n})_{ij} > 0$ then $d \geq 1$, hence

$$\deg(A^m)_{ij},\ \deg(A^n)_{jj} > 0$$

and therefore by assumption

$$\kappa_1(A^m)_{ij},\ \kappa_1(A^n)_{jj} \in \mathcal{P}_+$$

which implies $\kappa_1(A^{m+n})_{ij} \in \mathcal{P}_+$ by Lemma 4.1. Thus we have shown $m + n \in E_{ij}$.
(iii) This is proved analogously as (ii).
(iv) By (ii) the set E_{ii} is additively closed. Together with (i) this implies that the family of sets is suitable.
(v) Note that $E_{ij} \neq \emptyset$ for all $i, j \in [r]$. Thus (a) is clear by (ii), and (b) follows from (ii) and (iii) by

$$s + n + t = (s + n) + t \in E_{ij} + E_{ji} \subseteq E_{ii}.$$

(vi) Using (iv) and (v) we see that $\{E_{ij}\}_{i,j\in[r]}$ is a suitable aperiodic and properly irreducible family of subsets of $\mathbb{N}$. Then our assertion is clear by Theorem 7.4. □

For convenience we introduce the following notion.

Definition 5.15. The integer $n \in \mathbb{N}$ is called good for the matrix $A \in \mathcal{P}[X]^{r\times r}$ if for all $i, j \in [r]$ the following implication holds:

$$\deg(A^n)_{ij} > 0 \implies \operatorname{slc}(A^n)_{ij} \in \mathcal{P}_+. \tag{5.2}$$

Note that in case $\deg(\mathrm{A}) = 0$ all $n \in \mathbb{N}$ are good for A. Clearly, the existence of a good integer for A clearly does not imply the

Lemma 5.16. *If $m \in \mathbb{N}$ is good for the matrix $A \in \mathcal{P}[X]^{r\times r}$ then tm is good for A for all $t \in \mathbb{N}_{>0}$.*

Proof. Clearly, we need only consider the case $\deg(A) > 0$. Now, we proceed by induction on t exploiting Lemma 4.1. $\square$

We are now in a position to state a final result in this direction.

Theorem 5.17. *Let R be a unital commutative semiring of characteristic* 0, $\mathcal{P}_+$ *a preprime of* R, $0 \notin \mathcal{P}_+$, $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$, $r \in \mathbb{N}_{>0}$ *and* $A \in \mathcal{P}[X]^{r\times r}$. *The following statements are equivalent.*

(i) *A is $\mathcal{P}_+[X]$-primitive.*

(ii) *There exists some $N \in \mathbb{N}_{>0}$ such that for all $n \geq N$ we have*

$$A^n > 0 \qquad \textit{and} \qquad \deg(A^n)_{ij} \geq \deg(A) \qquad (i,j \in [r]).$$

(iii) *A is strongly $\mathcal{P}_+[X]$-irreducible and strongly $\mathcal{P}_+[X]$-aperiodic and there exists some $N \in \mathbb{N}_{>0}$ which is good for A.*

Proof. Let $d = \deg(A)$.
(i) $\implies$ (ii) Let $m \in \mathbb{N}_{>0}$ such that $A^m > 0$ and set $N = 2m(m+1)$. By Lemma 5.10 we have

$$\deg(A^N)_{ij} \geq m\, d \geq d \qquad (i,j \in [r]).$$

Clearly, A^N is 'positive' as it is the product of 'positive' factors:

$$A^N = \underbrace{A^m \cdots A^m}_{2(m+1)\text{ factors}}.$$

By Lemma 5.11 for all $j \in [r]$ there is some $i \in [r]$ such that $\kappa_0(A_{ij}) > 0$, and an application of Lemma 5.12 concludes the proof.
(ii) $\implies$ (iii) Obviously, A is strongly $\mathcal{P}_+[X]$-irreducible. For all $\in [r]$ the crucial set for the determination of $\operatorname{sper}_i(A)$ contains $\mathbb{N}_{\geq N}$, hence $\operatorname{sper}_i(A) = 1$, and A is strongly $\mathcal{P}_+[X]$-aperiodic. Clearly, N is good for A.
(iii) $\implies$ (i) By Lemma 5.14 (vi) we find some $m \in \mathbb{N}_{>0}$ such that for all $n \geq m$ and $i,j \in [r]$ we have $\kappa_0(A^n)_{ij} \in \mathcal{P}_+$, $\deg(A^n)_{ij} \geq \min\{1, d\}$ and

$$\deg(A^n)_{ij} > 0 \implies \kappa_1(A^n)_{ij} \in \mathcal{P}_+.$$

In case $d = 0$ the assertion is clear by Theorem 2.6, therefore we assume $d > 0$ and choose an integer t with $M = tN \geq m$. Thus, by the above, Lemma 5.16 and Definition 5.15 we find for all $i,j \in [r]$

$$\kappa_0(A^M)_{ij} \in \mathcal{P}_+, \qquad \deg(A^M)_{ij} > 0, \qquad \kappa_1(A^M)_{ij} \in \mathcal{P}_+, \qquad \operatorname{slc}(A^M)_{ij} \in \mathcal{P}_+,$$

which means that $(A^M)_{ij}$ is primitive by Theorem 4.3. Now, A^M is primitive by Theorem 5.7, and Remark 2.2 (ii) concludes the proof. $\square$

6. An alternative primitivity criterion for polynomial matrices

In this section we introduce a different tool to describe primitivity. Having in mind Remark 5.4 (i) we no longer study the behavior of the coefficient matrices (see Section 5.1) $\kappa_k(A)$ $(k > 1)$, but turn to some additional matrix which we think to better control the two highest coefficients of the entries of the polynomial matrix A.

6.1. **Almost linear periodic sequences in monoids.**

In this subsection we let $(\mathcal{M}, \circ)$ be a commutative monoid with neutral element e. For simplicity we write

$$a^n = \underbrace{a \circ \cdots \circ a}_{n\text{ factors}} \qquad (a \in \mathcal{M}, n \in \mathbb{N}_{>0})$$

and $a^0 = e$.

Definition 6.1. (cf. [9, Definition 2.3, 2.4]) The sequence $a^\star = (a_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{M}$ is called almost linear periodic if there are $N \in \mathbb{N}, p \in \mathbb{N}_{>0}$ and $q \in \mathcal{M}$ such that for every $n > N$

$$a_{n+p} = a_n \circ q^p. \tag{6.1}$$

In this case the smallest number $N \in \mathbb{N}$ with the property that there are $p \in \mathbb{N}_{>0}$ and $q \in \mathcal{M}$ such that (6.1) holds for every $n > N$ is called the linear defect of $a^\star$, and we write $N = \operatorname{ldef} a^\star$. The minimal number $p \in \mathbb{N}_{>0}$ such that there is some $q \in \mathcal{M}$ such that (6.1) holds for every $n > \operatorname{ldef} a^\star$ is called the linear period of $a^\star$, and we write $p = \operatorname{lper} a^\star$. Finally, an element q with (6.1) is called a linear factor of $a^\star$. In case q is unique we write $q = \operatorname{lfac} a^\star$.

We are now aiming at some kind of divisibility of elements of a monoid by positive integers. Clearly, this goal cannot be achieved for arbitrary monoids.

Definition 6.2. We say that the monoid $(\mathcal{M}, \circ)$ has property $(\star)$ if for every $a, b \in \mathcal{M}$ and $n \in \mathbb{N}_{>0}$ the equality $a^n = b^n$ implies $a = b$.

Remark 6.3. If the monoid $(\mathcal{M}, \circ)$ is cancellative it need not have property $(\star)$ (e.g., take the monoid of nonzero real numbers together with the usual multiplication).

For the remainder of this subsection we assume that $(\mathcal{M}, \circ)$ has property $(\star)$. We define the following relation on $\mathcal{M} \times \mathbb{N}_{>0}$:

$$(a, n) \sim (b, m) \iff a^m = b^n.$$

Lemma 6.4. (i) *$\sim$ is an equivalence relation on $\mathcal{M} \times \mathbb{N}_{>0}$.*

(ii) *The pair $(\widehat{\mathcal{M}}, \circ)$ consisting of the set of equivalence classes of $\mathcal{M} \times \mathbb{N}_{>0}$ and the binary operation*

$$[a, n] \circ [b, m] := [a^m \circ b^n, nm]$$

is a commutative monoid with neutral element $[e, 1]$; here we write $[a, n]$ for the class of (a, n). Furthermore, it is divisible[7] *and has property $(\star)$.*

(iii) *For every $a \in \mathcal{M}$ and $n \in \mathbb{N}_{>0}$ we have*

$$[a, n]^n = [a, 1].$$

(iv) *The map $\iota : \mathcal{M} \to \widehat{\mathcal{M}}$ given by $a \mapsto [a, 1]$ defines a monoid monomorphism.*

(v) *If $\mathcal{M}$ is divisible then $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are isomorphic.*

Proof. This can easily be checked. $\square$

In view of Lemma 6.4 we tacitly treat $\mathcal{M}$ as a submonoid of $\widehat{\mathcal{M}}$. In particular, this implies that the sequence $(a_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{M}$ is almost linear periodic if there are $N \in \mathbb{N}$, $p \in \mathbb{N}_{>0}$ and $q \in \widehat{\mathcal{M}}$ such that (6.1) is satisfied for all $n > N$.

Lemma 6.5. *Let $(\mathcal{M}, \circ)$ be a commutative cancellative monoid with property $(\star)$.*

(i) *$\widehat{\mathcal{M}}$ is cancellative.*

(ii) *Let $a^\star = (a_n)_{n\in\mathbb{N}}$ be an almost linear periodic sequence in $\mathcal{M}$. Then there is a uniquely determined $q \in \mathcal{M}$ with*

$$a_{n+p} = a_n \circ q^{\operatorname{lper} a^\star}$$

for every $n > \operatorname{ldef} a^\star$, and we write $q = \operatorname{lfac} a^\star$.

Proof. Using Lemma 6.4 this can easily be checked. $\square$

We close this subsection by an example of a monoid which will play a decisive role in Section 6.4.

[7] The monoid $(\mathcal{M}, \circ)$ is called divisible if for any $y \in \mathcal{M}$ and any $n \in \mathbb{N}_{>0}$ there is a unique $x \in \mathcal{M}$ such that $x^n = y$. For reasons of analogy we write $x = \frac{y}{n}$ (cf. [9, Definition 2.1]).

Example 6.6. $((\mathbb{Q}_{\geq 0} \times \{0,1\}) \setminus \{(0,1)\}, \circ, \leq)$ with

$$(x,a) \circ (y,b) = (x+y, \max\{a,b\}), \qquad (x,y \in \mathbb{Q}_{\geq 0},\ a,b \in \{0,1\}),$$

is a commutative linearly ordered divisible monoid where we order $\mathbb{Q}_{\geq 0} \times \{0,1\}$ lexicographically. Furthermore, this monoid has property $(\star)$, however, it is not cancellative (e.g., we have $(1,1) \circ (1,0) = (1,1) \circ (1,1)$). However, notice that

$$(x,a) \circ (z,0) = (y,b) \circ (z,0) \implies (x,a) = (y,b) \, .$$

6.2. **Almost linear periodic matrices over dioids.** Let $\mathcal{D} = (\mathcal{M}, \oplus, \otimes)$ be a commutative dioid[8] in the sense of [1, Def. 4.1, p. 154] with neutral elements ε and e, respectively[9]. It is well known that the set $\mathcal{M}^{r\times r}$ of $r \times r$ matrices over $\mathcal{M}$ yields a dioid where addition and multiplication of matrices and multiplication of matrices by elements of $\mathcal{M}$ are defined in the usual way.

Lemma 6.7. (i) *$\mathcal{D}$ is a commutative unital semiring of characteristic zero with idempotent sum (i.e., $x \oplus x = x$ for all $x \in \mathcal{D}$).*
(ii) *If $\mathcal{D}$ is entire*[10] *then $\mathcal{D} \setminus \{\varepsilon\}$ is a preprime of $\mathcal{D}$.*
(iii) *Let $\mathcal{D}$ be entire and $B \in \mathcal{M}^{r\times r}$ with $B_{ij} \neq \varepsilon$ for some $i,j \in [r]$. Then B is irreducible if and only if B is $\mathcal{M} \setminus \{\varepsilon\}$-irreducible.*

Example 6.8. ([9, Definition 2.1]) Let $(G, +, \leq)$ be a commutative linearly ordered divisible group and $G^\star = G \cup \{\varepsilon\}$ where $\varepsilon \notin G$ is a new element which satisfies $\varepsilon \leq x$ and $\varepsilon + x = x + \varepsilon = \varepsilon$ for all $x \in G^\star$. In other words, $(G^\star, +)$ is a monoid and ε is minimal w. r. t. $\leq$ and absorbing[11] w. r. t. $+$. We call the dioid $(G^\star, \oplus, \otimes)$ with $\oplus = \max$ and $\otimes = +$ the max-plus algebra generated by G. We observe that the monoid $(G, \otimes)$ is cancellative and enjoys property $(\star)$, and that the groups $(G, +)$ and $(\hat{G}, \otimes)$ are isomorphic (see Section 6.1). Furthermore, the max-plus algebra $(G^\star, \oplus, \otimes)$ is entire with idempotent sum. The natural order (see [1, Definition 4.11, p. 155]) on this dioid coincides with the linear order on the monoid $(G^\star, +)$.

The most important example of dioids of this type is the standard max-plus algebra[12]

$$\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \max, +)$$

generated by the real numbers with the usual addition and order relation.

We follow as closely as possibly the exposition given in [9].

Definition 6.9. (cf. [9, Definition 2.3, 2.4])

(i) The sequence $B^\star = (B_n)_{n\in\mathbb{N}}$ of matrices in $\mathcal{M}^{r\times r}$ is called almost linear periodic if for all $i,j \in [r]$ the sequence

$$(B^\star)_{ij} = ((B_n)_{ij})_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}$$

is almost linear periodic. In this case, the numbers

$$\operatorname{lper} B^\star := \operatorname{lcm} \{\operatorname{lper}(B^\star)_{ij} \ : \ i,j \in [r]\}$$

and

$$\operatorname{ldef} B^\star := \max \{\operatorname{ldef} (B^\star)_{ij} \ : \ i,j \in [r]\}$$

are called the linear period and the linear defect, respectively, of $B^\star$. Further, a matrix $Q \in \mathcal{M}^{r\times r}$ given by linear factors Q_{ij} of the sequences $((B_n)_{ij})_{n\in\mathbb{N}}$ $(i,j \in [r])$ is called

[8] By abuse of notation we sometimes use the symbol $\mathcal{D}$ instead of $\mathcal{M}$ and tacitly imply the use of the two binary operations $\oplus$ and $\otimes$.

[9] Clearly, $\mathcal{D}$ is a semiring, however, $\mathcal{D}$ cannot always be embedded into a ring (see the discussion in [1, p. 210]). Lots of examples are given in [11, Section 2.3]).

[10] The dioid $(\mathcal{M}, \oplus, \otimes)$ is entire if for all $x, y \in \mathcal{M}$ the equation $x \otimes y = \varepsilon$ implies $x = \varepsilon$ or $y = \varepsilon$ (see [1, Definition 4.11, p. 155]). Instead of "entire" one may find the expression "without zero divisors" in the literature.

[11] The element a of the monoid $(\mathcal{M}, \circ)$ is called absorbing if for any $x \in \mathcal{M}$ we have $a \circ x = a$.

[12] A comprehensive introduction to max-plus algebra is found in [1], applications are discussed in e.g. [5], [3]. Sometimes these semirings are also called tropical (see [19], [26]).

a linear factor matrix of $B^\star$. $i,j \in [r]$ If for each $i,j \in [r]$ there is a unique linear factor Q_{ij} we write

$$\text{lfac}\,(B^\star) = Q\ .$$

Let us now fix a matrix $B \in \mathcal{M}^{r\times r}$.

Definition 6.10. (cf. [9, Definition 2.3, 2.4]) We say that the matrix B is almost linear periodic if the sequence $(B^n)_{n\in\mathbb{N}}$ is almost linear periodic. In this case we simply write

$$\text{ldef}\,(B) = \text{ldef}\,((B^n)_{n\in\mathbb{N}}), \qquad \text{lper}\,(B) = \text{lper}\,((B^n)_{n\in\mathbb{N}}).$$

If there is a unique linear factor we write lfac $(B) =$ lfac $((B^n)_{n\in\mathbb{N}})$.

From now on we further assume that the monoid $(\mathcal{M}\setminus\{\varepsilon\}, \otimes)$ has the property $(\star)$.

Lemma 6.11. *Let $B \in \mathcal{M}^{r\times r}$ be irreducible.*

(i) *For all $i \in [r], n \in \mathbb{N}_{>0}$ there is some $j \in [r]$ such that $(B^n)_{ij} \neq \varepsilon$.*
(ii) *For all $j \in [r], n \in \mathbb{N}_{>0}$ there is some $i \in [r]$ such that $(B^n)_{ij} \neq \varepsilon$.*
(iii) *Let B be almost linear periodic and $\lambda(B) \in \widehat{\mathcal{M}}$ be a linear factor of B. Then $\lambda(B) \neq \varepsilon$, and if $\widehat{\mathcal{M}\setminus\{\varepsilon\}}$ is cancellative then $\lambda(B)$ is unique.*

Proof. This can easily be checked using Lemmata 6.7 and 2.4. □

Let us now continue the study of the graph associated to a given matrix (see Section 2.1). We denote by $|\pi|$ the length of the path[13] π in $\mathcal{G}(B)$ and by $W^{(n)}(i,j)$ the (possibly empty, but finite) set of all paths of length n from i to j. The set $W^{(0)}(i,i)$ consists of the empty path (of length 0) which both starts and terminates at i. For convenience we set

$$W(i,j) = \bigcup_{n\in\mathbb{N}_{>0}} W^{(n)}(i,j) \qquad \text{and} \qquad \Pi(B) = \bigcup_{i,j\in[r]} W(i,j)\ .$$

Let $\pi = (i_0, i_1, \ldots, i_n)$ be a path of positive length n in $\mathcal{G}(B)$. We call

$$v(\pi) = B_{i_0,i_1} \otimes \cdots \otimes B_{i_{n-1},i_n}.$$

the weight of π. For $\pi \in W^{(0)}(i,i)$ we set

$$v(\pi) = e\ .$$

We are interested in the function

$$\overline{v} : \Pi(B) \to \widehat{\mathcal{M}}$$

given by

$$\overline{v}(\pi) = [v(\pi), |\pi|]\ .$$

We call $\overline{v}(\pi)$ the average (or mean) weight of π. As we shall see below the mean weight of π reflects the "real" size of the path π. In particular, we are interested in the cycles of $\mathcal{G}(B)$, and therefore we denote by

$$C_B = \bigcup_{i\in[r]} \bigcup_{n\in\mathbb{N}_{>0}} W^{(n)}(i,i)$$

the set of cycles of positive lengths in $\mathcal{G}(B)$.

For the remainder of this section we further assume that $\leq$ is a linear order on the set $\widehat{\mathcal{M}\setminus\{\varepsilon\}}$ with the following property: For all $c, c' \in C_B$ we have

$$[v(c)\otimes v(c'), |c| + |c'|] \leq \max\{\overline{v}(c), \overline{v}(c')\}\ . \tag{6.2}$$

Lemma 6.12. (i) *Let $i \in [r], L \in \mathbb{N}_{>0}$ and $\pi \in W^{(L)}(i,i)$ with*

$$\overline{v}(\pi) = \max\left\{\overline{v}(\rho)\ :\ \rho \in \bigcup_{n=1}^{L} W^{(n)}(i,i)\right\}\ .$$

Then there is a subpath ρ of π with the following properties.

[13] We use the terminology of Lind – Marcus [16, Definition 2.2.11].

(a) $\rho \in W(i,i)$
(b) $|\rho| \leq r$
(c) $\overline{v}(\rho) = \overline{v}(\pi)$

(ii) *Assume that either for all* $i \in [r]$ *there is some* $j \in [r]$ *with* $B_{ij} \neq \varepsilon$ *or for all* $j \in [r]$ *there is some* $i \in [r]$ *with* $B_{ij} \neq \varepsilon$. *Then* $C_B \neq \emptyset$.

(iii) *If* $C_B \neq \emptyset$ *then there is a cycle* c *in* $\mathcal{G}(B)$ *of positive length at most* r *with the property*

$$\overline{v}(c) = \max\{\overline{v}(\rho) \ : \ \rho \in C_B\} \ .$$

In this case we call $\lambda(B) := \overline{v}(c)$ *the maximal cycle mean weight*[14] *of* B.

(iv) *If* B *is irreducible then* $\lambda(B) \in \widehat{\mathcal{M} \setminus \{\varepsilon\}}$.

Proof. (i) We may assume $L > r$ because otherwise there is nothing to prove. Let $\pi = (i_0, i_1, \ldots, i_L)$ with $i_0 = i_L = i$. Choose $k \in [L]$ minimal with $i_k = i$ and pick a subpath ρ of $(i_0, i_1, \ldots, i_k)$ which starts at i and terminates at i and which does not intersect itself. Then clearly $\rho \in W(i,i)$ and $|\rho| \leq r$. Let now τ be the path which extends ρ to π. By (6.2) we have

$$\overline{v}(\pi) = [v(\rho) \otimes v(\tau), |\rho| + |\tau|] \leq \max\{\overline{v}(\rho), \overline{v}(\tau)\} \ ,$$

hence

$$\overline{v}(\pi) = \max\{\overline{v}(\rho), \overline{v}(\tau)\}$$

by our choice of π. Therefore ρ or τ is a cycle of positive length less than L and equal $\overline{v}$-value as π. If necessary we apply the same procedure to this cycle instead of π. After finitely many steps we arrive at a cycle with the required properties.

(ii) Let us first assume that for all $i \in [r]$ there is some $j \in [r]$ with $B_{ij} \neq \varepsilon$. Then we can define a function on $[r]$ by

$$\iota(i) := \min\{j \in [r] \ : \ B_{ij} \neq \varepsilon\} \, .$$

The iterates of ι give a path in the finite set of vertices $[r]$, hence there must be some $k < m$ with

$$\iota^k(i) = \iota^m(i).$$

Thus we can construct a path of positive length from $\iota^k(i)$ to $\iota^m(i)$, i.e., an element of C_B. The second part of the statement is shown analogously.

(iii) Clear by (i).

(iv) As $C_B \neq \emptyset$ this follows from (iii). □

In case of a max-plus algebra the asymptotic behavior of the sequence of powers of an irreducible matrix is periodic and completely determined by the lengths and average weights of its cycles of maximal cycle mean weight[15] (see [9, Theorem 3.1]). We aim at exploiting this result for our purposes here.

Following [9, p. 169] we say that the vertices $i, j \in [r]$ are highly connected (and write $i \equiv_h j$) if i, j are contained in a cycle c with maximal cycle mean weight, i.e., $\overline{v}(c) = \lambda(B)$. It is easy to check that $\equiv_h$ defines a symmetric and transitive relation on the set of vertices of $\mathcal{G}(B)$. Note that in general high connectivity is not reflexive[16] (see Example 6.21 (i) below).

The subgraphs of $\mathcal{G}(B)$ induced by the classes of $\equiv_h$ are called highly connected components of $\mathcal{G}(B)$. A highly connected component in $\mathcal{G}(B)$ is called trivial if it contains no cycle of positive length with cycle mean weight equal to $\lambda(B)$. The set of all such components is denoted by $\mathcal{C}(B)$. Clearly, $\mathcal{K} \in \mathcal{C}(B)$ is not necessarily strongly connected. The high period of $\mathcal{K} \in \mathcal{C}(B)$ is defined by

$$\text{hper}\,(\mathcal{K}) := \gcd\ \{|c| \ : \ c \text{ cycle in } \mathcal{K},\ \overline{v}(c) = \lambda(B)\} \, ,$$

if $\mathcal{K}$ is nontrivial, and $\text{hper}\,(\mathcal{K}) = 0$, otherwise.

---

[14] Cf. [9, p. 169].

[15] From the viewpoint of applications of the classical max-plus algebra the (unique) cycle of maximal cycle mean weight is the 'slowest' cycle and eventually imposes its 'speed' on the whole system (see e.g. [3, p. 17]).

[16] The reflexive extension of this relation is an equivalence relation on the set of vertices of $\mathcal{G}(B)$ (cf. [23, p. 34]), but we do not need this here.

In the following sections we concentrate on a seemingly new dioid which is more adapted to our goals explained in the introduction.

6.3. **Description and properties of a particular dioid over the integers.** We introduce a dioid which describes the behavior of the two highest coefficients of the entries of the matrix A. More precisely, the elements of our dioid consist of pairs where the first component keeps book of the degree and the second component tells us whether or not the respective slc-coefficient is 'positive' or not. Clearly, in order to make statements about $\mathcal{P}_+[X]$-primitivity of polynomial matrices more information is needed as we shall see below.

On the set

$$\mathcal{D} := \big((\mathbb{N} \cup \{-\infty\}) \times \{0,1\}\big) \setminus \{(-\infty,1),(0,1)\}$$

we introduce two binary operations:

$$(n,a) \oplus (m,b) = (\max\{n,m\}, \delta_+(n,a,m,b)), \qquad \text{and} \qquad (n,a) \otimes (m,b) = (n+m, \delta_\times(n,a,m,b)),$$

where the functions $\delta_+, \delta_\times \ : \ \mathcal{D} \times \mathcal{D} \to \{0,1\}$ are defined as follows: First, $\delta_+(n,a,m,b) = 1$ if one of the following three conditions is satisfied:

(i) $\max\{|m-n|, a, b\} = 1$,
(ii) $n > m+1$ and $a = 1$,
(iii) $m > n+1$ and $b = 1$,

otherwise $\delta_+(n,a,m,b) = 0$. Second, $\delta_\times(n,a,m,b) = \max\{a,b\}$ if $n, m \in \mathbb{N}$, and $\delta_\times(n,a,m,b) = 0$, otherwise. One may imagine that the first component stands for the arrival time of supply while the second component stands for its successiveness. If the second component is 1 then we think that the supply is stable, at least locally. The above definition of $\oplus$ models the situation that two unstable supplies arrive at consecutive time units, say n and $n+1$, and then we judge that the supply line becomes stable. It is also assumed that once the supply line becomes stable then it will remain stable.

Lemma 6.13. (i) $(\mathcal{D}, \oplus, \otimes)$ *is a commutative dioid in the sense of* [1, Def. 4.1, p. 154] *with neutral elements* $\varepsilon = (-\infty, 0)$ *and* $e = (0,0)$*, respectively.*
(ii) *The element* ε *is absorbing for* $\otimes$.
(iii) $(\mathcal{D}, \oplus, \otimes)$ *is a commutative unital semiring of characteristic* 0*, and*

$$\mathcal{Q} := \{(n,1) \in \mathcal{D} \ : \ n \in \mathbb{N}_{>0}\}$$

is a preprime contained in the preprime $\mathcal{Q} \cup \{\varepsilon\}$.
(iv) *The monoid* $(\mathcal{D} \setminus \{\varepsilon\}, \otimes)$ *has property* $(\star)$*, and* $(\widehat{\mathcal{D} \setminus \{\varepsilon\}}, \otimes)$ *is isomorphic to* $(\mathbb{Q}_{\geq 0} \times \{0,1\}) \setminus \{(0,1)\}, \circ)$ *(see Section 6.1) where the isomorphism is given by*

$$[(n,a), m] \mapsto \big(\frac{n}{m}, a\big) \ .$$

Proof. This can easily be checked. □

Remark 6.14. $\mathcal{D}$ cannot be embedded into a max-plus algebra because otherwise we would have $\alpha \oplus \beta = \max\{\alpha, \beta\}$, hence in particular

$$(1,1) = (1,0) \oplus (0,0) \in \{(1,0),(0,0)\}$$

which is impossible[17]. The natural order (see [1, Theorem 4.28, p. 160]) on $\mathcal{D}$ is not linear (e.g., the elements $(1,1)$ and $(2,0)$ are incomparable).

For the remainder of this subsection we let $B \in \mathcal{D}^{r \times r}$.

Remark 6.15. If B is irreducible then B need not be $\mathcal{Q}$-irreducible (e.g., take $B = (e)$). However, if B is $\mathcal{Q}$-irreducible then B is irreducible.

[17] $\mathcal{D}$ satisfies the weak (but not the strong) stabilization condition (e.g., see [15] and the references therein), but we do not use this fact here.

By our conventions introduced above for the path π of positive length we have

$$\overline{v}(\pi) = \left(\frac{v(\pi)_1}{|\pi|},\ v(\pi)_2\right) \in \mathbb{Q}_{\geq 0} \times \{0,1\}\ ,$$

and using the lexicographical order on $\mathbb{Q}_{\geq 0} \times \{0,1\}$ (cf. Example 6.6) the paths of positive lengths in the digraph $\mathcal{G}(B)$ are linearly ordered by their mean weights. In particular, the cycles of positive lengths in $\mathcal{G}(B)$ enjoy property (6.2) and are linearly ordered by the cycle mean weights.

Lemma 6.16 below which is an analogue of [9, Lemma 2.1] describes the (i,j)-entry of the powers of B by the weight of paths from vertex i to vertex j.

Lemma 6.16. *(cf.* [9, Lemma 2.1], [11, p. 11]*) Let $i,j \in [r]$, $n \in \mathbb{N}_{>0}$ and $w \in W^{(n)}(i,j)$.*

(i) *If*

$$\overline{v}(w) = \max\left\{\overline{v}(\pi)\ :\ \pi \in W^{(n)}(i,j)\right\}\ . \tag{6.3}$$

we have

$$((B^n)_{ij})_1 = v(w)_1 \quad \text{and} \quad ((B^n)_{ij})_2 \geq v(w)_2\ .$$

(ii) *If $v(w)_1 = ((B^n)_{ij})_1$ and $v(w)_2 = 1$ then w satisfies (6.3), and we have $(B^n)_{ij} = v(w)$.*

Proof. (i) is clear by the definitions, and (ii)is immediate by (i). □

Remark 6.17. Note that for $r > 1$ we might have

$$(B^n)_{ij} \neq \max\left\{v(\pi)\ :\ \pi \in W^{(n)}(i,j)\right\},$$

e.g. for

$$B = \begin{pmatrix} (2,0) & (2,0) \\ e & (1,1) \end{pmatrix}$$

we have two paths of length 2 from 1 to 2, namely $\pi = (1,1,2)$ and $\rho = (1,2,2)$. We have

$$v(\pi) = (4,0) > (3,1) = v(\rho),$$

but $(B^2)_{12} = (4,1)$. Observe that $(B^2)_{12} \notin v(W^{(2)}(1,2))$, i.e., there is no path of length 2 from 1 to 2.

We denote by $F \in \mathbb{R}_{\max}^{r\times r}$ $(S_n \in \{0,1\}^{r\times r}$, respectively$)$ the matrix of first components of B (the matrix of second components of B^n, respectively), more precisely

$$F_{ij} = \left(B_{ij}\right)_1, \qquad (S_n)_{ij} = \left((B^n)_{ij}\right)_2 \qquad (i,j \in [r]).$$

Clearly,

$$F^n = (B^n)_1 \qquad (n \in \mathbb{N})$$

where addition and multiplication on the left hand side is performed in $\mathbb{R}_{\max}$ (see Example 6.8).

The graphs $\mathcal{G}(F)$ and $\mathcal{G}(B)$ coincide, and every cycle of maximum mean weight in $\mathcal{G}(B)$ is a cycle of maximum mean weight in $\mathcal{G}(F)$. However, the converse does not hold as the following example shows.

Example 6.18. For

$$B = \begin{pmatrix} (1,0) & e \\ e & (1,1) \end{pmatrix}$$

we have

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

thus $1 \to 1$ is a cycle of maximum mean weight in $\mathcal{G}(F)$, but not in $\mathcal{G}(B)$.

We now collect some important relations between the properties of B and its component matrices.

Proposition 6.19. *Let $B \in \mathcal{D}^{r\times r}$ be irreducible.*

(i) *F is almost linear periodic. We have* $\mathrm{lfac}\,(F)_{ij} = \lambda(B)_1 > 0$ *for all $i, j \in [r]$ and*

$$p := \mathrm{lper}\,(F) = \mathrm{lcm}\,\{\mathrm{hper}\,\mathcal{K} \;:\; \mathcal{K} \in \mathcal{C}(F),\ \mathcal{K}\ \textit{nontrivial}\}\,,$$

and $\lambda(B)$ and p can be computed in $O(r^3)$ time. If $\mathcal{K}_1, \dots, \mathcal{K}_t$ are the highly connected components of $\mathcal{G}(F)$ which contain the nontrivial highly connected components of $\mathcal{G}(B)$ then $\mathrm{lcm}\,\{\mathrm{hper}\,\mathcal{K}_k \;:\; k \in [t]\}$ *divides* $\mathrm{lcm}\,\{\mathrm{hper}\,\mathcal{K} \;:\; \mathcal{K} \in \mathcal{C}(B),\ \mathcal{K}\ \textit{nontrivial}\}$.

(ii) *B is almost linear periodic and $\lambda(B)_1$ is the first component of a linear factor of B.*

(iii) *Let us assume that the sequence of the second components of the powers of B is ultimately constant, i.e., there is some $M \in \mathbb{N}$ with*

$$S_n = S_M \qquad (n \geq M). \tag{6.4}$$

Then we have

$$\mathrm{ldef}\,(F) \leq \mathrm{ldef}\,(B) \leq \max\{M, \mathrm{ldef}\,(F)\}\,.$$

Proof. (i) Clear by [9, Theorem 3.1 and Theorem 3.6] and our remarks above.
(ii) Fix $i, j \in [r]$. It suffices to prove that there is some $N > \mathrm{ldef}\,(F)$ such that

$$(B^{n+p})_{ij} = (B^n)_{ij}\ (\lambda(B)_1, 0)^p$$

for all $n > N$. By (i) this equation holds for the first components, thus we are left to show

$$(B^{n+p})_{ij2} = (B^n)_{ij2}\ .$$

Assume first that we have

$$(B^{n+p})_{ij2} = 0$$

for all $n > \mathrm{ldef}\,(F)$. Then we set $N = p + \mathrm{ldef}\,(F)$, and we are done.

Now, we suppose that

$$(B^{m+p})_{ij2} = 1$$

for some $m > \mathrm{ldef}\,(F)$. Then we set $N = p + m$. By the proof of [9, Theorem 3.1] there is a path $\pi \in W(i, j)$ and a cycle c of maximum mean weight in $\mathcal{G}(F)$ such that

$$v(\pi c^t)_1 = (B^{m+p})_{ij1}\ .$$

We distinguish two cases.

First, let $v(\pi c^t)_2 = 1$. Then clearly $v(\pi c^s)_2 = 1$ for all $s \geq t$, and our assertion follows by Lemma 6.23.

Second, let $v(\pi c^t)_2 = 0$. Then there must be some path $\rho \in W(i, j)$ with $|\rho| = m + p$ and $v(\rho)_1 = v(\pi c^t)_1 - 1$. Let $n \geq m$. It suffices to establish $(B^{n+p})_{ij2} = 1$. Again inspecting the proof of [9, Theorem 3.1] we find $s \geq t$ with

$$v(\pi c^s)_1 = (B^{n+p})_{ij1}\ .$$

Obviously,

$$\left|\rho c^{s-t}\right| = m + p + (s - t)\,|c| = |\pi| + t\,|c| + (s - t)\,|c| = |\pi| + s\,|c| = |\pi c^s|\,,$$

hence $v(\rho c^{s-t})$ is one of the summands occurring in the computation of $(B^{n+p})_{ij}$. Analogously we find

$$v(\rho c^{s-t})_1 = v(\pi c^s)_1 - 1$$

yielding $(B^{n+p})_{ij2} = 1$ by adding $v(\pi c^s)$ and $v(\rho c^{s-t})$.
(iii) This is clear by (i). ☐

Remark 6.20. For small r Lemma 6.12 (ii) allows an easy calculation of $\lambda(B)$ provided B is irreducible (e.g., see Example 6.21). In general, Karp's method [1, Chapter 2] allows the calculation of $(\lambda(B))_1$.

The following simple example illustrates our results.

Example 6.21. We show that the maximal cycle mean weight is not necessarily a linear factor of an irreducible matrix. For the matrix

$$B = \begin{pmatrix} e & \beta \\ e & \varepsilon \end{pmatrix}$$

with $\beta = (1, 0)$ we have

$$F = \begin{pmatrix} 0 & 1 \\ 0 & -\infty \end{pmatrix}.$$

The first few powers of B are

$$B^2 = \begin{pmatrix} \gamma & \beta \\ e & \beta \end{pmatrix},\ B^3 = \begin{pmatrix} \gamma & \zeta \\ \gamma & \beta \end{pmatrix},\ B^4 = \begin{pmatrix} \zeta & \zeta \\ \gamma & \zeta \end{pmatrix},\ B^5 = \begin{pmatrix} \zeta & \omega \\ \zeta & \zeta \end{pmatrix},\ B^6 = \begin{pmatrix} \omega & \omega \\ \zeta & \omega \end{pmatrix}.$$

where we set $\gamma = (1, 1), \zeta = (2, 1)$ and $\omega = (3, 1)$. Thus B is $\mathcal{Q}$-primitive of exponent 4, and F is $\mathbb{R}_{>0}$-primitive of exponent 3. Further we note that the sequence of second components of the powers of B is ultimately constant, and $M = 4$ is the minimal integer which satisfies (6.4).

By induction we easily check

$$B^n = \begin{cases} \begin{pmatrix} (n-3,1) & (n-2,1) \\ (n-3,1) & (n-3,1) \end{pmatrix} & (n \text{ odd}), \\ \\ \begin{pmatrix} (n-3,1) & (n-3,1) \\ (n-4,1) & (n-3,1) \end{pmatrix} & (n \text{ even}) \end{cases} \tag{6.5}$$

for $n \geq 5$.

There is one cycle of length 1, and we have $\overline{v}(1 \to 1) = e$. For the cycle c (of length 2) given by $1 \to 2 \to 1$ we have $v(c) = \beta$, and there are two other cycles of length 2. More explicitly, we have

$$\overline{v}(1 \to 1 \to 1) = e, \qquad \overline{v}(2 \to 1 \to 2) = (\frac{1}{2}, 0),$$

hence we have found

$$\text{lfac}\ (F) = \lambda(F) = \frac{1}{2}, \qquad \lambda(B) = (\frac{1}{2}, 0)$$

by Proposition 6.19. Further, our calculations above and (6.5) yield ldef $(F) = 1$.

The cycle c yields the only nontrivial highly connected component $\mathcal{K}$, and we have

$$\text{lper}\ (F) = \text{hper}\ (\mathcal{K}) = 2$$

by Lemma 6.12. Using (6.5) we easily check

$$B^{n+2} = B^n \otimes \lambda(B)^2 \qquad (n > 3), \tag{6.6}$$

therefore B is almost linear periodic with ldef $(B) \leq 3$.

Setting $\mu = (\frac{1}{2}, 1)$ and using (6.6) we find

$$B^{n+2} = B^n \otimes \mu^2 \qquad (n > 1), \tag{6.7}$$

hence we even have ldef $(B) \leq 1$. Further, this shows ldef $(B) = 1$, because otherwise there would be $\nu \in \mathbb{Q}_{\geq 0} \times \{0, 1\}$ and $p \in \mathbb{N}_{>0}$ such that

$$B^{n+p} = B^n \otimes \nu^p \qquad (n > 0),$$

in particular

$$(B^{1+p})_{22} = B_{22} \otimes \nu^p = \varepsilon \otimes \nu^p = \varepsilon,$$

contradicting (6.7) (note that $p = 1$ is impossible by our explicit calculations in the beginning of this example). Further we have lper $(B) \leq 2$ which yields lper $(B) = 2$ because otherwise we had

$$B^{n+1} = B^n \otimes \nu \qquad (n > 1),$$

with some $\nu \in \mathbb{Q}_{\geq 0} \times \{0, 1\}$, hence

$$\beta = (B^{2+1})_{22} = (B^2)_{22} \otimes \nu = \beta \otimes \nu,$$

and then $\nu = \beta$ which is impossible. We determine lfac $(B) = \mu$, because

$$B^{n+1} = B^n \otimes (1,0) \qquad (n > 1)$$

yields

$$(1,1) = e \otimes \mu^2 = (B^2)_{21} \otimes \mu^2 = (B^2)_{21} \otimes \nu^2 = \nu^2$$

and then $\nu = \mu$. Finally we observe that in this example the last inequality in (6.4) is strict.

6.4. **Application to polynomial matrices.** Now we connect our usual setting with the dioid $\mathcal{D}$ constructed in the previous section. Notice that $\mathcal{D}$ does not depend on the semiring R.

For $f \in R[X]$ we set $\delta(f) = 1$ if $\mathrm{slc}(f) \in \mathcal{P}_+$, and $\delta(f) = 0$, otherwise. Further, we define a map $s : R[X] \to \mathcal{D}$ by

$$s(f) = (\deg(f), \delta(f)),$$

thus the first entry stands for the degree and the second entry takes care of the slc-coefficient of f.

Lemma 6.22. *Let $f, g \in \mathcal{P}[X]$.*

(i) *$s(f+g) = s(f) \oplus s(g)$ and $s(fg) = s(f) \otimes s(g)$, thus s is a semiring epimorphism.*

(ii) *If $n \in \mathbb{N}, p \in \mathbb{N}_{>0}$ and $q := s(f)$ then*

$$s(f^{n+p}) = s(f^n) \otimes \underbrace{q \otimes \cdots \otimes q}_{p\ factors},$$

thus $s(f)$ is almost linear periodic with linear defect 0 and linear period 1. Further, $s(f)$ is a linear factor of $s(f)$, and in case $\delta(f) = 0$ we have lfac $(s(f)) = s(f)$.

Proof. This can easily be checked. □

Lemma 6.23. *Let $A, B \in \mathcal{P}[X]^{r\times r}$.*

(i) *We have*

$$s(AB) = s(A) \otimes s(B),$$

thus s is a monoid epimorphism.

(ii) *If A is $\mathcal{P}_+[X]$-primitive and $\deg(A) > 0$ then $s(A)$ is $\mathcal{Q}$-primitive.*

(iii) *If A is $\mathcal{P}_+[X]$-irreducible then $s(A)$ is irreducible.*

Proof. (i) This can easily be checked.
(ii) By Theorem 5.17 there is some $N \in \mathbb{N}_{>0}$ such that for all $n \geq N$ we have

$$A^n > 0 \qquad \text{and} \qquad \deg(A^n)_{ij} \geq \deg(A) \qquad (i, j \in [r]).$$

Thus $s(A)$ is $\mathcal{Q}$-primitive by (i).
(iii) Clear by (i). □

We aim at showing a different criterion for $\mathcal{P}_+[X]$-primitivity.

Theorem 6.24. *Let R be a unital commutative semiring of characteristic 0, $\mathcal{P}_+$ a preprime of R, $0 \notin \mathcal{P}_+$ and $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$. Let $r \in \mathbb{N}_{>0}$ and $A \in \mathcal{P}[X]^{r\times r}$ with $\deg(A) > 0$. Then A is $\mathcal{P}_+[X]$-primitive if and only if the following conditions hold:*

(i) *$\kappa_0(A)$ is $\mathcal{P}_+$-primitive and $\kappa_1(A)$ is nonzero.*

(ii) *$s(A)$ is $\mathcal{Q}$-irreducible and $\mathcal{Q}$-aperiodic.*

Proof. We first show sufficiency. Our assumption (i) and Lemma 5.1 assure that there is some $N \in \mathbb{N}_{>0}$ so that $\kappa_0(A^n)_{i,j}$ and $\kappa_1(A^n)_{i,j}$ are 'positive' for all $i, j \in [r]$ and all $n \geq N$.

By Theorem 2.6 there is some $M \in \mathbb{N}_{>0}$ with

$$s(A^n)_{ij} \in \mathcal{Q} \qquad (i, j \in [r],\ n \geq M).$$

Thus, for some $n \geq \max\{N, M\}$ we find that $(A^n)_{ij}$ is $\mathcal{P}_+[X]$-primitive for all $i, j \in [r]$. Therefore Theorem 5.7 yields our assertion.

Conversely, if

$$(A^n)_{ij} \in \mathcal{P}_+[X] \qquad (i, j \in [r])$$

is satisfied for some $n \in \mathbb{N}_{>0}$ then (i) holds, and $s(A)$ is $\mathcal{Q}$-primitive by Lemma 6.23, whence (ii) follows by Theorem 2.6. □

Now we state our main result, namely a relation between the primitivity of the matrix A of positive degree and the behavior of the matrix sequence $(s(A^n))_{n\in\mathbb{N}}$. To simplify the notation we write $\lambda(A) := \lambda(s(A))$.

Theorem 6.25. *Let R be a unital commutative semiring of characteristic* 0*, $\mathcal{P}_+$ a preprime of R, $0 \notin \mathcal{P}_+$ and $\mathcal{P} = \mathcal{P}_+ \cup \{0\}$, $r \in \mathbb{N}_{>0}$ and $A \in \mathcal{P}[X]^{r\times r}$ with* $\deg(A) > 0$*. Then the following statements are equivalent:*

(i) *A is $\mathcal{P}_+[X]$-primitive.*
(ii) *A has the following properties:*
  (a) *$\kappa_0(A)$ is $\mathcal{P}_+$-primitive.*
  (b) *$\kappa_1(A)$ is nonzero.*
  (c) *A is $\mathcal{P}_+[X]$-irreducible.*
  (d) *$s(A)$ is almost linear periodic, $(\lambda(A)_1, 1)$ is a linear factor of $s(A)$ and $\lambda(A)_1 > 0$.*
(iii) *A has the following properties:*
  (a) *$\kappa_0(A)$ is $\mathcal{P}_+$-primitive.*
  (b) *$\kappa_1(A)$ is nonzero.*
  (c) *$s(A)$ is almost linear periodic, $s(A)$ admits a linear factor in $\mathbb{Q}_{>0} \times \{1\}$.*
(iv) *A has the following properties:*
  (a) *$\kappa_0(A)$ is $\mathcal{P}_+$-primitive.*
  (b) *$\kappa_1(A)$ is nonzero.*
  (c) *$s(A)$ is $\mathcal{Q}$-primitive.*

Proof. (i) $\Longrightarrow$ (ii) By Theorem 5.17 there is some $N \in \mathbb{N}_{>0}$ such that for all $n \geq N$ we have

$$(A^n)_{ij} \in \mathcal{P}_+[X], \qquad \text{and} \qquad \deg(A^n)_{ij} \geq \deg(A) \qquad (i,j \in [r]),$$

thus $(a), (b), (c)$ are satisfied by Lemma 5.1 and the definition of $\mathcal{P}_+[X]$-irreducibility. By Proposition 6.19 $s(A)$ is almost linear periodic. Now, (d) follows from Lemma 6.23 (iii) and Proposition 6.19 for n sufficiently large (note that we can replace the second component of $\lambda(A)$ by 1 for large n).
(ii) $\Longrightarrow$ (iii) Obvious.
(iii) $\Longrightarrow$ (iv) By assumption there are $N,\, p,\, a,\, m \in \mathbb{N}_{>0}$ such that

$$s(A)^{n+p} = s(A)^n\, (\frac{a}{m}, 1)^p \qquad (n > N),$$

hence

$$s(A^n)_{ij} \in \mathcal{Q} \qquad (n > N + p,\ i,j \in [r]),$$

i.e., $s(A)$ is $\mathcal{Q}$-primitive.
(iv) $\Longrightarrow$ (i) By (b) we can pick $\nu, \mu \in [r]$ such that $\kappa_1(A_{\nu\mu}) > 0$. By (a) and (c) there exists $N \in \mathbb{N}_{>0}$ such that

$$\operatorname{lc}((A^n)_{ij}),\quad \operatorname{slc}((A^n)_{ij}),\quad \kappa_0((A^n)_{ij}) > 0 \qquad (n > N,\ i,j \in [r]). \tag{6.8}$$

Now, let $n > 2(N+2)$, $N < \ell < n - N - 1$ and fix $i,j \in [r]$. Then clearly

$$\kappa_0((A^{n-1-\ell})_{i\nu})\, \kappa_1(A_{\nu\mu})\, \kappa_0((A^\ell)_{\mu j}) \ > \ 0\,.$$

By Lemma 5.1 we have

$$\kappa_1((A^n)_{ij}) = \sum_{\sigma=0}^{n-1} \left(\kappa_0(A^{n-1-\sigma})\, \kappa_1(A)\, \kappa_0(A^\sigma)\right)_{ij}\,. \tag{6.9}$$

One of the summands occurring on the right hand side is

$$\left(\left(\kappa_0(A^{n-1-\ell})\left(\kappa_1(A)\, \kappa_0(A^\ell)\right)\right)\right)_{ij} = \sum_{\rho=1}^{r} \kappa_0((A^{n-1-\ell})_{i\rho}) \left(\kappa_1(A)\, \kappa_0(A^\ell)\right)_{\rho j}\,.$$

The sum on the right hand side contains

$$\kappa_0((A^{n-1-\ell})_{i\nu}) \left(\kappa_1(A)\, \kappa_0(A^\ell)\right)_{\nu j} = \kappa_0((A^{n-1-\ell})_{i\nu}) \sum_{\tau=1}^{r} \kappa_1(A_{\nu\tau}) \left(\kappa_0(A^\ell)\right)_{\tau j}\,.$$

Thus on the right hand side of (6.9) we find the 'positive' summand

$$\kappa_0((A^{n-1-\ell})_{i\nu})\,\kappa_1(A_{\nu\mu})\,(\kappa_0(A^\ell))_{\mu j}\,,$$

and we have shown $\kappa_1((A^n)_{ij}) > 0$. Together with (6.8) we infer from Theorem 4.3 that $(A^n)_{ij}$ is primitive. Finally, an application of Theorem 5.7 concludes the proof. □

Remark 6.26. We loosely gather some observations which shed some more light on our settings.

(i) Let $A \in \mathcal{P}[X]^{r\times r}$, $\deg(A) > 0$ and $s(A)$ irreducible. If A is $\mathcal{P}_+[X]$-primitive then $s(A^{\mathrm{ldef}(A)+\mathrm{lper}(A)}) \in \mathcal{Q}^{r\times r}$. But the converse does not hold as the following example shows. Let $A = X^2 + X \in \mathcal{P}[X]$. Then $s(A) = (2,1) \in \mathcal{Q}$, $\mathrm{ldef}(A) = 0$, $\mathrm{lper}(A) = 1$, but A is not $\mathcal{P}_+[X]$-primitive.

(ii) If

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$s(A) = \begin{pmatrix} \varepsilon & e \\ e & \varepsilon \end{pmatrix}$$

is irreducible, and we have $\lambda(A) = e$, $\mathrm{lper}\,(A) = \mathrm{ldef}\,(A) = 1$. In the natural order of $\mathcal{D}$ (see [1, Theorem 4.28]) we have

$$s(A^2)_{12} = \varepsilon \;<\; e = s(A)_{12}.$$

Thus, the sequence $((s(A)_{ij})^n)_{n\in\mathbb{N}}$ is not necessarily increasing. Certainly $s(A)$ is not $\mathcal{Q}$-primitive.

(iii) For the non-$\mathcal{P}_+[X]$-primitive matrix

$$A = \begin{pmatrix} 1 & X \\ 1 & 0 \end{pmatrix}$$

we know that $s(A)$ is $\mathcal{Q}$-primitive of exponent 4, and we have $\mathrm{ldef}\,(A) = 1$ (see Example 6.21).

(iv) It seems that for $r > 1$ [15, Theorem 7] does not hold in full generality: The irreducible matrix

$$\begin{pmatrix} e & (1,0) \\ e & \varepsilon \end{pmatrix} \in \mathcal{D}^{2\times 2}$$

over the dioid $\mathcal{D}$ defined in Section 6.3 does not have an eigenvalue.

In view of [9] we conjecture that our main result Theorem 6.25 can be extended to an algorithm. Furthermore, we suspect that some other classical results on powers of matrices with nonnegative entries can similarly be derived under the prerequisites of Theorem 2.6, but delay details to subsequent work.

7. Appendix

We collect some facts on families of integer sets which are used in our considerations thereby giving a formal treatment of the characterization of primitive real matrices in the vein of Pták [21] or Holladay – Varga [14].

Definition 7.1. The family $\{E_{ij}\}_{i,j\in[r]}$ of subsets of $\mathbb{N}$ is called suitable if it enjoys the following properties:

(i) For all $i \in [r]$ the set E_{ii} is additively closed.
(ii) For all $i, j \in [r]$ with $i \neq j$ one has $0 \notin E_{ij}$.

For $i \in [r]$ we let $\mathrm{per}\,(i)$ be the greatest common divisor of the set $E_{ii} \cap \mathbb{N}_{>0}$ if this set is non-void, and $\mathrm{per}\,(i) = \infty$, otherwise (see [25]).

Lemma 7.2. *Let $\{E_{ij}\}_{i,j\in[r]}$ be a suitable family of subsets of $\mathbb{N}$ with the following property: For all $i,j \in [r]$ with $i \neq j$ there are $s \in E_{ij}$ and $t \in E_{ji}$ with*

$$s + n + t \in E_{ii}$$

for all $n \in E_{jj} \cup \{0\}$. Then

$$\mathrm{per}\,(i) = \mathrm{per}\,(j) \in \mathbb{N}_{>0}$$

for all $i \neq j$.

Proof. Let $i \neq j$. Clearly, $s + t \in E_{ii} \cap \mathbb{N}_{>0}$, hence $\mathrm{per}\,(i) \in \mathbb{N}_{>0}$, and analogously $\mathrm{per}\,(j) \in \mathbb{N}_{>0}$. Thus,

$$s + t, (s + t) + n \in E_{ii} \cap \mathbb{N}_{>0}$$

for all $n \in E_{jj} \cap \mathbb{N}_{>0}$ which implies that $\mathrm{per}\,(i)$ divides every $n \in E_{jj} \cap \mathbb{N}_{>0}$. Therefore,

$$\mathrm{per}\,(i) \mid \mathrm{per}\,(j).$$

Interchanging the roles of i and j our assertion follows. □

Definition 7.3. Let $\mathcal{E} = \{E_{ij}\}_{i,j\in[r]}$ be a suitable family of subsets of $\mathbb{N}$.

(i) $\mathcal{E}$ is called aperiodic if

$$\gcd\left(\mathrm{per}\,(1), \dots, \mathrm{per}\,(r)\right) = 1.$$

(ii) $\mathcal{E}$ is called properly irreducible if for all $i,j \in [r]$ the following properties are satisfied:
   (a) There exists $m \in E_{ij}$ such that $m + n \in E_{ij}$ for all $n \in E_{jj}$.
   (b) If $i \neq j$ then there are $s \in E_{ij}$ and $t \in E_{ji}$ with

$$s + n + t \in E_{ii}$$

   for all $n \in E_{jj} \cup \{0\}$.

Theorem 7.4. *Let $\{E_{ij}\}_{i,j\in[r]}$ be a suitable family of subsets of $\mathbb{N}$ and*

$$E = \bigcap_{i,j\in[r]} E_{ij}.$$

Then the following statements are equivalent.

(i) *There exists some $m \in \mathbb{N}_{>0} \cap E$ such that $n \in E$ for all $n \geq m$.*
(ii) *$\mathcal{E}$ is properly irreducible and aperiodic.*

Proof. Let $D_i = E_{ii} \cap \mathbb{N}_{>0}$ and assume that $\mathcal{E}$ is properly irreducible and aperiodic. Then $\mathrm{per}\,(i) = 1$ for all i: This is trivial for $r = 1$, and for $r > 1$ it is clear by Lemma 7.2. By [25, Lemma A.3] for all i there is some $n_i \in D_i$ such that $n \in D_i$ for all $n \geq n_i$. Pick $k_{ij} \in E_{ij}$ according to property (a) of the definition of proper irreducibility and put $m = N + K$ where

$$N = \max\{n_1, \dots, n_r\}, \qquad K = \max\{k_{ij} \;:\; i,j \in [r]\}.$$

For $n \geq m$ and $i,j \in [r]$ we have

$$n - k_{ij} \geq m - k_{ij} = N + (K - k_{ij}) \geq N \geq n_j$$

which yields $n - k_{ij} \in E_{jj}$ and

$$n = k_{ij} + (n - k_{ij}) \in E_{ij}.$$

On the other hand, observe $m, m+1 \in D_i$, hence $\mathrm{per}\,(i) = 1$ for all $i \in [r]$. Thus, $\mathcal{E}$ is aperiodic, and the proper irreducibility of $\mathcal{E}$ is obvious. □

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