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ON THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS OF TRIANGLE GROUPS

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§0. Introduction

Denote $J_d(z)$ the absolute invariant of the Hecke group G_d. Then J_d has the following Fourier expansion at $i\infty$:

$$J_d(z) = \sum_{n=-1}^{\infty} a_n r^n q^n,$$

where $a_n \in Q$, $r \in R$ and $q = \exp\left(\frac{\pi i z}{\cos(\pi/d)}\right)$.
The value r is algebraic if and only if $d = 3, 4, 6$ and ∞ ([3], [6]). These results can be extended to the case of fuchsian triangle groups and the expansion at an elliptic fixed point ([7], [8]). In this paper we consider the ratio of the value r's when there is an inclusion relation of groups. In §2 we prove using purely algebraic method that the ratio is algebraic and etc. In the remaining section we put into concrete this result in the case of triangle groups. Especially in this case, some power of the ratio belongs to the imaginary quadratic field.

§1. Notation and results

Let q be an indeterminant, and K be some subfield of the complex number field C. The *quasi K-rational power series of style* r is the formal power series of the form

$$\sum_{n \geq \ell} a_n r^n q^n \quad (a_n \in K,\ r \in C^* = C - \{0\},\ \ell \in Z).$$

The *quasi K-rational vector space of style* r is the vector space over C spaned by these series. The style r of a power series is determined up to an equivalence relation:

$$(r_1/r_2)^s \in K \qquad \text{----------} \ (1)$$

for some $s \in Z$.
Let F be a subfield of C containing K, and $c_1, c_2, \cdots, c_t$ be the complex

numbers. We say that $\{c_1, c_2, \cdots, c_t\}$ is F-independent over K if the property (P) is satisfied for all $d_i \in K$ $(i = 1, \cdots, t)$.

$$\text{(P)} \qquad \sum_{i=1}^{t} d_i c_i \in F \quad \text{then} \quad d_i = 0 \quad \text{for} \quad i = 1, \cdots\cdot, t.$$

We can now state the main theorem.

THEOREM 1. *Let* V *be the quasi K-rational vector space of style* r_1, *and* f *be an element of* V. *Suppose that* f *is the quasi K-rational infinite power series of style* r_2. *Then the ratio of the styles* $\gamma = r_1/r_2$ *is algebraic over* K, *and* f *is a linear combination of the basis of quasi* K*-rational power series of style* r_1 *over* $K(\gamma)$. *Moreover there are distinct non negative integers* l_0 $(= 0)$, $l_1, \cdots\cdots, l_m$ $(0 \leq m \leq$ dim $V)$, *and infinite numbers of* n *such that* $\{\gamma^{n-l_0}, \gamma^{n-l_1}, \cdots\cdots, \gamma^{n-l_m}\}$ *is not* K*-independent.*

We can take the value m not larger than the maximum number of power series in the quasi K-rational basis whose leading coefficients a_ℓ is 0. In §4 we will consider automorphic forms which has real axis as the natural boundary. In this case, the condition of *infinite* series is naturally satisfied. The conclusion of this theorem is rather complicated, but if the following conjecture holds, we can rewrite the theorem in a better style.

CONJECTURE. *Assume that* $\gamma \in C$ *has the last properties of Theorem* 1 *then* γ^t *is an algebraic number of degree* $m + 1$ *over* K *for some natural number* t. *Exchanging indices, we can write* $l_i = l \cdot i$ $(i = 0, \cdots, m)$.

The style r for a quasi K-rational power series is determined by the equivalence relation (1). The style of the quasi K-rational *vector space* is determined by the following theorem.

THEOREM 2. *Let* V *be the quasi K-rational vector space whose style is taken in two ways as* r_1, r_2. *If* V *has at least one infinite power series, then there exists some natural number* s *such that*

$$(r_1/r_2)^s \in K.$$

Choose basis of V *of the form*

$$\Sigma\, a_{n,k} r_1^n q^n \quad (k = 1, 2, \cdots, s\colon\ s = \dim V).$$

If the vector $(a_{n,1}, a_{n,2}, \cdots\cdots\cdots, a_{n,s})$ *is non zero for all* n, *then the number* s

can be taken not larger than dim V.

§2. The proof of Theorem 1

Let

$$\sum_{n \geq \ell} a_{n,k} r_1^n q^n \quad (k = 1, 2, \cdots\cdots, s = \dim V)$$

be the basis of V, where $a_{n,k} \in K$, $r_1 \in C^*$ and $\ell \in Z$. By the assumption, we have

$$\begin{aligned} f &= \sum_{n \geq \ell} c_n r_2^n q^n \\ &= \sum_{k=1}^{s} d_k \left(\sum_{n \geq \ell} a_{n,k} r_1^n q^n \right) \in V. \end{aligned}$$

So

$$c_n r_2^n = r_1^n \sum_{k=1}^{s} d_k a_{n,k} \qquad (n \geq \ell). \qquad \cdots\cdots\cdots\cdots (2)$$

Put $\gamma = r_1/r_2$, $D = (d_1, d_2, \cdots\cdots\cdots, d_s)$ and

$$a_n = {}^t(a_{n,1}, a_{n,2}, \cdots\cdots\cdots\cdots, a_{n,s}).$$

Then (2) is written in the form

$$c_n = \gamma^n D \cdot a_n. \qquad \cdots\cdots\cdots\cdots (3)$$

So

$$c_n = \gamma^n (D \cdot P) \cdot (P^{-1} \cdot a_n)$$

for $P \in GL_s(K)$. We can change basis of V by this method in order to get the assertion. At first we say that d_i $(i = 1, \cdots, s)$ can be taken in $K(\gamma)$. Assume $d_1 \notin K(\gamma)$. If $d_1, d_2, \cdots\cdots, d_t$ are $K(\gamma)$-independent and $d_1, d_2, \cdots\cdots\cdots, d_{t+1}$ are not $K(\gamma)$-independent, then we replace d_{t+1} with

$$d_{t+1} + \sum_{i=1}^{t} h_i d_i \quad (h_i \in K).$$

Thus we are able to think that d_{t+1} belongs to $K(\gamma)$ from the start. Repeating this argument we get

$$\begin{cases} d_1, d_2, \cdots\cdots\cdots, d_t \text{ are } K(\gamma)\text{-independent:} \\ d_{t+1}, \cdots\cdots\cdots, d_s \text{ belong to } K(\gamma), \text{ where } t \geq 1. \end{cases}$$

From (2) we have

$$\frac{c_n}{\gamma^n} - \sum_{k=t+1}^{s} d_k\, a_{n,k} = \sum_{k=1}^{t} d_k\, a_{n,k} \in K(\gamma).$$

Thus $a_{n,k} = 0$ for $k = 1, \cdots\cdots, t$. This is a contradiction. So we get $d_k \in K(\gamma)$ for $k = 1, 2, \cdots, s$. Using similar arguments we can assume

$$\begin{cases} \gamma^{\ell} d_1, \gamma^{\ell} d_2, \cdots\cdots\cdots\cdots, \gamma^{\ell} d_t \text{ are } K\text{-independent ;} \\ d_{t+1} = d_{t+2} = \cdots\cdots\cdots = d_s = 1/\gamma^{\ell} . \end{cases}$$

Without losing generality, we can assume $t < s$. Define

$$T_n = \left\{ (g_1, g_2, \cdots\cdots, g_t, g^*) \in K^{t+1} \;\middle|\; g^* \gamma^{n-\ell} + \sum_{k=1}^{t} \gamma^n d_k g_k \in K \right\}$$

and

$$S_{n,e} = \left\{ (g_1, g_2, \cdots\cdots, g_{t-e+1}) \in K^{t-e+1} \;\middle|\; (g_1, g_2, \cdots\cdots, g_t, g^*) \in T_n \right\}.$$

We define $n_1, n_2, \cdots$ by induction. By the definition we know

$$S_{\ell,1} = \{0\}.$$

Let n_1 be the smallest number of n such that $S_{n,1} \neq \{0\}$. We may assume $g_t \neq 0$ so that we can replace d_t by

$$d_t + g_t^{-1} \sum_{k=1}^{t-1} d_k g_k + g_t^{-1} g^* \gamma^{-\ell},$$

and multiply some number in K^* : we can put $d_t = 1/\gamma^{n_1}$. If $n_1, n_2, \cdots\cdots, n_w$ are defined and $d_t = 1/\gamma^{n_1}, d_{t-1} = 1/\gamma^{n_2}, \cdots, d_{t-w+1} = 1/\gamma^{n_w}$, then we may assume $S_{n,w} = \{0\}$ for $n = \ell, \ell+1, \cdots\cdots, n_w - 1$, and $S_{n_w, w+1} = \{0\}$. Since we have choosed the basis of V, there is a number n such that $S_{n,w+1} \neq \{0\}$ if $w+1 \leq t$. Let n_{w+1} be the smallest number of these. Then we may put $d_{t-w} = 1/\gamma^{n_{w+1}}$, according to the same argument. Thus we may consider that $d_1 = 1/\gamma^{n_t}$, $d_2 = 1/\gamma^{n_{t-1}}$, $\cdots\cdots\cdots$, $d_t = 1/\gamma^{n_1}$. There are infinite numbers of n such that $n > n_w$ and $a_n \neq 0$, because f is an infinite power series. This concludes the proof.

§3. The proof of Theorem 2

Let

$$\sum_{n \geq \ell} a_{n,k}\, r_1^n\, q^n, \quad \sum_{n \geq \ell} b_{n,k}\, r_2^n\, q^n \quad (k = 1, 2, \cdots\cdots, s = \dim V)$$

be two quasi K-rational basis of V whose styles r_1, r_2 respectively. Put

$$\sum_{n\geq\ell} b_{n,k}\, r_2^n\, q^n = \sum_{k=1}^{s} d_{j,k} \sum_{n\geq\ell} a_{n,k}\, r_1^n\, q^n,$$

$$\gamma = r_1/r_2,\quad D = (d_{j,k}),$$

$$\boldsymbol{a}_n = {}^t(a_{n,1}, a_{n,2}, \cdots\cdots\cdots\cdots, a_{n,s}),$$

$$\boldsymbol{b}_n = {}^t(b_{n,1}, b_{n,2}, \cdots\cdots\cdots\cdots, b_{n,s}).$$

Then

$$\boldsymbol{b}_n = \gamma^n D \cdot \boldsymbol{a}_n . \qquad \text{-----------}(4)$$

So

$$P\boldsymbol{b}_n = \gamma^n (P\cdot D\cdot Q)\cdot(Q^{-1}\boldsymbol{a}_n) \qquad P, Q \in GL_s(K).$$

In this way we will change basis. Next lemma is well known (see [4] page 81).

Lemma. *Let* $\sum_{n\geq\ell} \xi_{n,k}\, q^n$ $(k = 1, \cdots\cdots, s)$ *be linearly independent formal power series over* C. *Put* $\Xi_n = {}^t(\xi_{n,1}, \xi_{n,2}, \cdots\cdots, \xi_{n,s})$, *then the vector space spaned by all* Ξ_n $(n = 1, 2, \cdots\cdots)$ *has rank* s.

Take $n_1, n_2, \cdots\cdots, n_s$ $(n_i \geq \ell)$ such that $\boldsymbol{a}_{n_1}, \boldsymbol{a}_{n_2}, \cdots\cdots, \boldsymbol{a}_{n_s}$ are linearly independent over C. Then from (4) we get

$$D\cdot(\boldsymbol{a}_{n_1}, \boldsymbol{a}_{n_2}, \cdots\cdots, \boldsymbol{a}_{n_s}) = (\gamma^{-n_1}\boldsymbol{b}_{n_1}, \gamma^{-n_2}\boldsymbol{b}_{n_2}, \cdots\cdots, \gamma^{-n_s}\boldsymbol{b}_{n_s}).$$

Put

$$P^{-1} = (\boldsymbol{b}_{n_1}, \boldsymbol{b}_{n_2}, \cdots\cdots, \boldsymbol{b}_{n_s}),\quad Q = (\boldsymbol{a}_{n_1}, \boldsymbol{a}_{n_2}, \cdots\cdots, \boldsymbol{a}_{n_s}).$$

Then

$$P\cdot D\cdot Q = \begin{bmatrix} \gamma^{-n_1} & & & \\ & \gamma^{-n_2} & & \\ & & \ddots & \\ & & & \gamma^{-n_s} \end{bmatrix}.$$

Since there are at least one infinite power series, there exists n such that $n > n_s$ and $\gamma^{n-n_k} \in K$ for some k $(k = 1, \cdots, s)$. This assures the first assertion of Theorem 2. Put

$$U = \{ \Xi \in C^s \mid D \cdot \Xi \in K^s \},$$

where $\Xi = {}^t(\xi_1, \xi_2, \cdots, \xi_s)$. U is the vector space over K. We define the linear map φ by

$$\begin{tikzcd}
\varphi : U \arrow[r] & K^s \\
\rotatebox[origin=c]{90}{$\in$} & \rotatebox[origin=c]{90}{$\in$} \\
\xi \arrow[r] & D \cdot \xi
\end{tikzcd}$$

As φ is injective, we get $\dim_K U \leq s$. Each $\gamma^n \boldsymbol{a}_n$ belongs to U. So $\gamma^\ell \boldsymbol{a}_\ell$, $\gamma^{\ell+1} \boldsymbol{a}_{\ell+1}, \cdots\cdots, \gamma^{\ell+s} \boldsymbol{a}_{\ell+s}$ are linearly dependent over K. There exist $(k_0, k_1, \cdots\cdots, k_s) \in K^{s+1} - \{0\}$ such that

$$\sum_{j=0}^{s} k_j \gamma^{\ell+j} \boldsymbol{a}_{\ell+j} = 0 .$$

We can find j $(j = 0, \cdots, s)$ such that $k_j \neq 0$, then choose i $(i = 1, \cdots, s)$ such that $a_{\ell+j,i} \neq 0$. Then

$$\sum_{j=0}^{s} k_j \gamma^j a_{\ell+j,i} = 0$$

gives the non trivial algebraic relation whose degree is not larger than s. This proves the second statement of Theorem 2.

§4. The ratio of styles in the case of fuchsian triangle groups

In this section we treat the special case of fuchsian triangle groups. For the precise notation, we refer to [8]. Let $\Delta = \Delta(p, q, r)$ be the triangle group whose signature is (p, q, r). If $1/p + 1/q + 1/r < 1$ then this group is realized and acts on the complex upper half plane H discontinuously. The fundamental domain of Δ is ABCD where ABC is the hyperbolic triangle, and ADC is the reflexion with respect to the geodesic AC. Denote $A_\Delta^{k,\nu}$ the space of holomorphic automorphic forms of Δ and of weight k, multiplier ν. Take $f \in A_\Delta^{k,\nu}$ then f is expanded at the elliptic point A of order p :

$$f(z) = (z - \bar{A})^{-k} \sum_{n \geq 0} a_n \left(\frac{z - A}{z - \bar{A}}\right)^n .$$

Ignoring $(z - \bar{A})^{-k}$, we know that $A_\Delta^{k,\nu}$ is the quasi rational vector space. The style of $A_\Delta^{k,\nu}$ depend only on the vertice A and Δ. Choosing good fundamental domain as Th 2 in [8], we write down this style value.

$$r(p;q,r) =$$

$$\frac{\Gamma(1+\frac{1}{p})\Gamma(\frac{1}{2}\left\{1-\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right\})\Gamma(\frac{1}{2}\left\{1-\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right\})}{\Gamma(1-\frac{1}{p})\Gamma(\frac{1}{2}\left\{1+\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right\})\Gamma(\frac{1}{2}\left\{1+\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right\})}\, s\ ,$$

where $s^2 = \dfrac{\cos(\epsilon-\frac{\pi}{p})\cos(\epsilon-\frac{\pi}{q})}{\cos(\epsilon)\cos(\epsilon-\frac{\pi}{r})}$, $\epsilon = \frac{\pi}{2}(\frac{1}{p}+\frac{1}{q}+\frac{1}{r})$.

We can easily check that

$$r(p;q,r) = r(p;r,q).$$

Assume $\Delta_1 = \Delta_1(p_1, q_1, r_1) \subset \Delta_2 = \Delta_2(p_2, q_2, r_2)$ and $A_1B_1C_1D_1$ be the fundamental domain of Δ_1 which is suitably located in the sense of Th 2 of [8]. That is to say, $A_1 = \sqrt{-1}$ and $B_1 = t\sqrt{-1}$ $(t > 1)$. We can't always assume that $A_2B_2C_2D_2$ is suitably located. Let ϕ be the natural covering map from $\Delta_1\backslash H$ to $\Delta_2\backslash H$, and assume $\phi(A_1) = A_2$. Of course $p_1 \mid p_2$. Denote θ $(0 \leq \theta \leq \pi)$ the angle of $B_2A_1B_1$. All inclusion relations of triangle groups are classified in [5]. So we can calculate the value θ in a straight forward way. After tedious calculations we know that

$$\cos(2p_2\theta) \in Q$$

for all inclusion relations. For example, in the case of Δ_1 (5,4,4) $\subset \Delta_2$(5,2,4), we get $\theta = \frac{\pi}{10}$. When we regard this relation as Δ_1(4,4,5) $\subset \Delta_2$(4,5,2), we get $\cos(4\theta) = \frac{3}{5}$. The rotation at A_1 and of angle θ causes small change of the style. Using the relation of [8] page 4, we see that the style is multiplied by $e^{\sqrt{-1}\theta}$. In all cases, the value $e^{\sqrt{-1}\theta}$ is algebraic. From Theorem 1, we see that the ratio $r(p_1;q_1;r_1)/r(p_2;q_2,r_2)$ is algebraic, because $A^{k,\nu}_{\Delta_1} \supset A^{k,\nu}_{\Delta_2}$ and $A^{k,\nu}_{\Delta_2}$ contains elements other than constant functions for sufficiently large k. If the conjecture of §1 is true, then some power of the ratio $r(p_1;q_1,r_1)/r(p_2;q_2,r_2)$ is of degree at most 7, because

$$\dim A^{k,\nu}_{\Delta_1} \leq \dim A^{k,\nu}_{\Delta_2} + 3 \leq 6,$$

for some k.

Thus we are interested in calculating these ratio of the styles.

Proposition. *Let* Δ_1, Δ_2 *be fuchsian triangle groups and* $\Delta_1 \subset \Delta_2$. *Then the ratio* $r(p_1;q_1,r_1)/r(p_2;q_2,r_2)$ *is given by the following table.*

(I) Normal case

$$\frac{r(p;p,p)}{r(p;3,3)} = 3^{-\frac{3}{2p}} \qquad \frac{r(p\ ;p,p)}{r(2p;2,3)} = 2^{\frac{1}{p}}\, 3^{-\frac{3}{2p}}$$

$$\frac{r(p\ ;q,q)}{r(2p;2,q)} = 2^{\frac{1}{p}} \qquad \frac{r(q;q,\ p)}{r(q;2,2p)} = 2^{-\frac{2}{q}}$$

(II) Non normal case

$$\frac{r(7;7,7)}{r(7;2,3)} = 2^{-\frac{6}{7}}\, 3^{-\frac{3}{7}}$$

$$\frac{r(7;7,2)}{r(7;2,3)} = 2^{-\frac{1}{2}}\, 3^{-\frac{3}{7}} \qquad \frac{r(2;7,7)}{r(2;3,7)} = 3^{-\frac{1}{2}}\, 7^{-\frac{1}{4}}$$

$$\frac{r(7;3,3)}{r(7;2,3)} = 2^{-\frac{6}{7}}\, 3^{-\frac{3}{14}} \qquad \frac{r(3;3,7)}{r(3;2,7)} = 2^{-1}\, 7^{-\frac{1}{6}}$$

$$\frac{r(8;8,4)}{r(8;2,3)} = 2^{-\frac{1}{2}}\, 3^{-\frac{3}{8}} \qquad \frac{r(4;8,8)}{r(8;2,3)} = 3^{-\frac{3}{8}}$$

$$\frac{r(8;8,3)}{r(8;2,3)} = 2^{-\frac{1}{4}}\, 3^{-\frac{1}{2}} \qquad \frac{r(3;8,8)}{r(3;2,8)} = 2^{-\frac{3}{2}}$$

$$\frac{r(9;9,9)}{r(9;2,3)} = 2^{-\frac{2}{3}}\, 3^{-\frac{1}{6}} \qquad \frac{r(9;9,9)}{r(3;2,9)} = 2^{-1}\, 3^{-\frac{5}{6}}$$

$$\frac{r(5;4,4)}{r(5;2,4)} = 2^{-1} \qquad \frac{r(4;4,5)}{r(4;2,5)} = 2^{-\frac{1}{2}}\, 5^{-\frac{1}{4}}$$

$$\frac{r(4p;4p,p)}{r(4p;\ 2,3)} = 2^{-\frac{1}{2p}}\, 3^{-\frac{3}{4p}} \qquad \frac{r(p;4p,4p)}{r(4\mathrm{p};2,\ 3)} = 2^{\frac{5}{2p}}\, 3^{-\frac{3}{4p}}$$

$$\frac{r(2p;2p,p)}{r(2p;\ 2,4)} = 2^{-\frac{2}{p}} \qquad \frac{r(p;2p,2p)}{r(2p;2,\ 4)} = 1$$

$$\frac{r(3p;3,p)}{r(3p;2,3)} = 2^{-\frac{2}{p}} \qquad \frac{r(p;3,3p)}{r(3p;2,3)} = 2^{-\frac{2}{p}}\, 3^{\frac{2}{p}}$$

$$\frac{r(3;3p,p)}{r(3;2,3p)} = 2^{-1}$$

$$\frac{r(2p;2,p)}{r(2p;2,3)} = 3^{-\frac{3}{2p}} \qquad \frac{r(p;2,2p)}{r(2p;2,3)} = 2^{\frac{3}{p}}\, 3^{-\frac{3}{2p}}$$

$$\frac{r(2;2p,p)}{r(2;2p,3)} = 3^{-\frac{1}{2}}$$

COROLLARY. *Let Δ_1, Δ_2 be fuchsian triangle groups and $\Delta_1 \subset \Delta_2$. We have*

$$(r(p_1;q_1,r_1)\,/\,r(p_2;q_2,r_2))^{2p_2} \in Q\,.$$

Prime factors which appear in the numerator and the denominator are the prime factors of $q_1r_1q_2r_2$.

REMARK. Consider the case $\Delta(5,4,4) \subset \Delta(5,2,4)$. As the elliptic point of order 4 of $\Delta(5,4,4)$ and the elliptic point of order 2 of $\Delta(5,2,4)$ are *not* identified by the covering map ϕ, it seems that we can't get the assertion of the corollary when we calculate $r(4;4,5)/r(2;4,5)$.

(The value becomes $\pi^{-\frac{3}{2}}\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{40})\,\Gamma(\frac{9}{40})$ up to algebraic factor.)

So we can get informations not only of the inclusion relation but also of the covering surface from this corollary.

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