On a certain sum of traces of Hecke operators

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§ 0. Introduction

Let Γ be a finitely generated fuchsian group of the first kind containing $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, $\mathbb{H}$ be the complex upper half plane and m be a non negative integer. Take a unitary representation χ of Γ of degree ν which satisfies $\chi\left(\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right)=(-1_{\nu})^m$. Denote by $\mathcal{L}^2_{\chi}(\Gamma\backslash\mathbb{H},m)$ the space of measurable functions from $\mathbb{H}$ to $\mathbb{C}^{\nu}$ satisfying

(1) $$\int_{\Gamma\backslash\mathbb{H}} f(z)\,{}^t\overline{f(z)}\,dz < \infty,$$

(2) $$f(\gamma\cdot z)\left(\frac{|cz+d|}{cz+d}\right)^m = \chi(\gamma)f(z),$$

for all $\gamma = \begin{pmatrix}* & *\\ c & d\end{pmatrix} \in \Gamma$. Put

$$\Delta_m = y^{-2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - \sqrt{-1}\, m\, y\frac{\partial}{\partial x}.$$

Then Δ_m acts on $\mathcal{L}^2_{\chi}(\Gamma\backslash\mathbb{H},m)$, and the spectral decomposition of this space is given by

$$\mathcal{L}^2_{\chi}(\Gamma\backslash\mathbb{H},m) = \left(\bigoplus_{\lambda}\mathcal{L}^2_{\chi}(\Gamma\backslash\mathbb{H},m,\lambda)\right)\oplus\mathcal{E},$$

where $\mathcal{L}^2_{\chi}(\Gamma\backslash\mathbb{H},m,\lambda)$ is the space of Maass wave forms of weight m, and $\mathcal{E}$ is the orthogonal complement. The eigenvalues are counted with multiplicities in the following way

$$\frac{m}{2}\left(\frac{m}{2}-1\right) \ge \lambda_1 \ge \lambda_2 \ge \cdots\cdots.$$

We define $\lambda = -1/4 - r^2$, $\rho = 1/2 + \sqrt{-1}\,r$. Then the Weyl-Selberg asymptotic formula is given by

$$N_{\Gamma}(T) - \frac{1}{4\pi}\int_{-T}^{T}\mathrm{tr}\left(\Phi'(1/2+\sqrt{-1}r)\,\Phi(1/2-\sqrt{-1}r)\right)dr$$

$$= \frac{\nu\,\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi}T^2 + O(T\log T),$$

where $N_{\Gamma}(T) = \sum_{|\rho|<T,\,\mathrm{Im}\,\rho>0} 1$, and $\Phi(s)$ is the scattering matrix of the Maass-Eisenstein series defined at the cusps of Γ (see [2] for the precise notation). When Γ is a congruence subgroup, we can see that the contribution of the scattering matrix is $O(T \log T)$. But in general, this might be false (see [7],[8]). The purpose of this note is to develop an analogue of this formula, using the Selberg trace formula for modular correspondences which was written down in [2]. Then we can get the asymptotic formula for a certain sum of traces of Hecke operators.

§ 1. The results

Take α from $SL(2,\mathbb{R})-(\pm 1)$ so that $\alpha^{-1}\Gamma\alpha$ is commensurable with Γ. Assume that χ is a unitary representation of degree ν of the group generated by Γ and α, which satisfies $\chi\left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right) = (-1_\nu)^m$. Define the Hecke operator acting on $\mathcal{L}^2_\chi(\Gamma\backslash\mathbb{H},m)$ by

$$T(\Gamma\alpha\Gamma)f(z) = \sum_\mu \chi(\alpha_\mu)f(\alpha_\mu^{-1}z)\left(\frac{|cz+d|}{cz+d}\right)^m,$$

where $\Gamma\alpha\Gamma = \bigcup_\mu \alpha_\mu \Gamma$ (disjoint) and $\alpha_\mu^{-1} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. Denote by $T(\Gamma\alpha\Gamma,\lambda_i)$ the restriction of $T(\Gamma\alpha\Gamma)$ on $\mathcal{L}^2_\chi(\Gamma\backslash\mathbb{H},m,\lambda_i)$.

Theorem

Put $N_{\Gamma\alpha\Gamma}(T) = \sum_{|\rho|<T,\, \mathrm{Im}\,\rho>0} \mathrm{tr}(\ T(\Gamma\alpha\Gamma,\lambda_i)\ )$. *Suppose that* Γ *has only one* Γ*-inequivalent cusp* ∞ *and the stabilizer of* ∞ *is generated by* $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *We also assume that* $\chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1_\nu$ *and* $\Gamma\alpha\Gamma = \Gamma\alpha^{-1}\Gamma$. *Then we have*

$$N_{\Gamma\alpha\Gamma}(T) - \frac{1}{4\pi}\int_{-T}^{T} \mathrm{tr}(\ W(1/2+\sqrt{-1}r)\Phi'(1/2+\sqrt{-1}r)\Phi(1/2-\sqrt{-1}r)\ )\,dr$$

$$= O(T \log T),$$

where $W(s) = \sum_{\tau=\begin{pmatrix} * & * \\ 0 & d \end{pmatrix},\ \tau \in \Gamma_\infty\backslash\Gamma\alpha\Gamma/\Gamma_\infty} \frac{(\mathrm{sgn}\ d)^m}{|d|^{2s}}\chi^{-1}(\tau)$. *Here we denote by* Γ_∞ *the stabilizer group of the cusp* ∞.

Remark 1. The assumptions on the cusps of Γ are not essential. But the assumption $\Gamma\alpha\Gamma = \Gamma\alpha^{-1}\Gamma$ seems to be necessary for our proof.

Remark 2. The summation in the definition of $W(s)$ is finite. So we see that $W(s)$ is entire and bounded in any vertical strip.

Remark 3. Comparing $N_{\Gamma}(T)$ with $N_{\Gamma\alpha\Gamma}(T)$, we notice that the right hand side of $N_{\Gamma}(T)$ is asymptotically larger than that of $N_{\Gamma\alpha\Gamma}(T)$. For $\Gamma=SL(2,\mathbb{Z})$ and $\alpha = \frac{1}{\sqrt{p}}\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, we see that the Eisenstein part on the left hand side is $O(T \log T)$ in both cases. Hence we have

$$N_{\Gamma\alpha\Gamma}(T) \;/\; N_{\Gamma}(T) \to 0,$$

when $T \to \infty$. This fact suggests there is much cancellation in the terms $\mathrm{tr}(\, T(\Gamma\alpha\Gamma,\lambda_i)\,)$.

§ 2. Analytic continuation of the Selberg trace formula

To prove the theorem, we use the Selberg trace formula for modular correspondences for the kernel function

$$h(r) = h(r,s) = \frac{2s-1}{r^2+(s-1/2)^2} - \frac{2s-1}{r^2+\beta^2},$$

with a sufficiently large positive constant β. The trace formula for the general kernel function was developed in [2]. So we employ the results of [2] freely.

Now we get the analytic continuation of the trace formula with respect to the valuable s. We can rewrite the trace formula in a product form

$$\Xi(s) = \Xi_{ell}(s)\, \Xi_{hyp_{(1)}}(s)\, \Xi_{hyp_{(2)}}(s)\, \Xi_{par}(s)\, \Xi_{Eis}(s),$$

for $Re(s) > \max(1/2, m/2)$ (cf. Fischer [3]). Here each term $\Xi_*(s)$ of the right hand side corresponds to the elliptic, hyperbolic$_{(1)}$, hyperbolic$_{(2)}$, parabolic conjugacy classes of $\Gamma\alpha\Gamma$ with respect to Γ. In other words,

$$\Xi_{ell}'(s)/\Xi_{ell}(s)$$

is the contribution of the elliptic conjugacy classes of the Selberg trace formula for $T(\Gamma\alpha\Gamma)$ and so on. $\Xi_{Eis}(s)$ corresponds to the contribution of the Eisenstein term of the Selberg trace formula. We denote by "hyperbolic$_{(1)}$" the hyperbolic conjugacy classes of $\Gamma\alpha\Gamma$ which fix hyperbolic fixed points of Γ and by "hyperbolic$_{(2)}$" the hyperbolic conjugacy classes which fix cusps. The singularity of $\Xi'(s)/\Xi(s)$ is given by

$$\sum_{\lambda_i} \frac{(2s-1)\ \mathrm{tr}(T(\Gamma\alpha\Gamma,\lambda_i))}{r_i^2+(s-1/2)^2}.$$

So we may write formally

$$\Xi(s) = \prod (-\lambda_i+s(s-1))^{\mathrm{tr}(T(\Gamma\alpha\Gamma,\lambda_i))}$$
$$= \det(-\Delta_m + s(s-1))^{T(\Gamma\alpha\Gamma)}.$$

Considering the case $\alpha=1$, which was excluded at the start, we see that $\Xi(s)$ is the functional determinant which is discussed recently by physicists (see [9],[11]).

In the following lemmas 1~3, we omit the β-term of the kernel function $h(r)$ because $\Xi'_{hyp_{(2)}}(s)/\Xi_{hyp_{(2)}}(s)$, $\Xi'_{ell}(s)/\Xi_{ell}(s)$, $\Xi'_{par}(s)/\Xi_{par}(s)$ are absolutely convergent without subtracting the β-term.

Lemma 1. *We have*

$$\frac{\Xi'_{hyp_{(2)}}(s)}{\Xi_{hyp_{(2)}}(s)} = \sum_{[P]_{hyp_{(2)}}} \Bigg[\frac{2(\mathrm{sgn}\ \mathrm{tr}P)^m\ \mathrm{tr}\ \chi(P)\ \log|c(P)|}{N\{P\}^{1/2} - N\{P\}^{-1/2}} N\{P\}^{-(s-1/2)}$$

$$+\ 2\ N\{P\}^{-(s-1/2)}\Big(\frac{\delta}{2s-1} + \sum_{k=1}^{\infty} \frac{N\{P\}^{-k} + (-1)^{m+k}}{k+2s-1}$$

$$-\sum_{k=1}^{[m/2]} \frac{2(m-2k+1)}{(2s-1)^2-(m-2k+1)^2}\Bigg)\Bigg],$$

where the first summation is finite and the infinite sum over k *is absolutely convergent for* $\mathrm{Re}(s)>0$. *Here* $N(P)$ *and* $c(P)$ *are defined as follows. Suppose that* P *is conjugate to* $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}$ *in* $SL(2,\mathbb{R})$ *with* $|\xi|>1$ *and* P *fixes* $\gamma_1\infty, \gamma_2\infty$ ($\gamma_i \in \Gamma$). *Then we denote by* $N(P)$ *the square of* ξ *and by* $c(P)$ *the* (2,1)*-element of* $\gamma_1^{-1}\gamma_2$. *We put*
$\delta = \begin{cases} 1 & (\text{ if } m \text{ is even }) \\ 0 & (\text{ if } m \text{ is odd }) \end{cases}.$

Lemma 2. *We have*

$$\Xi_{ell}(s) = \prod_{[R]} \prod_{\ell=0}^{r-1} \left[\Gamma\left(\frac{2s+2\ell+m}{2r}\right)^{e^{\sqrt{-1}\theta m}} \Gamma\left(\frac{2s+2\ell-m}{2r}\right)^{e^{-\sqrt{-1}\theta m}} \right]^{L(R,\ell)}$$

$$L(R,\ell) = \frac{\sqrt{-1}\; e^{\sqrt{-1}\theta(2\ell+1)} \operatorname{tr} \chi(R)}{2r^2 \sin\theta},$$

where $\Gamma(s)$ *is the gamma function. Here the first product* $\prod$ *extends over a system of representatives* R *of the elliptic conjugacy classes of* $\Gamma\alpha\Gamma$ *with respect to* Γ. *Suppose* R *is conjugate to the matrix* $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ *in* $SL(2,\mathbb{R})$. *Then we define by* r *the order of the centralizer group of* R *in* Γ.

Lemma 3. *Suppose that the set of parabolic elements in* $\Gamma\alpha\Gamma$ *which fix infinity is written in the form* $\bigcup_{\mu} \alpha_\mu \Gamma_\infty$ *(disjoint) and* $\chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1_\nu$. *Then we have*

$$\frac{\Xi'_{par}(s)}{\Xi_{par}(s)} = \sum_{\mu} \sum_{j=1}^{\nu} \exp(2\pi\sqrt{-1}\beta_{j\mu})\, I^*(\xi_j, v(\alpha_\mu))$$

$$I^*(\xi,v) = \frac{1}{2}\Bigg[\psi(s-m/2)+\psi(s+m/2)-2\psi(s)-2\gamma-\log 4 + \frac{2}{2s-1} - 2\psi(s+1/2)$$

$$-\psi(1-v)-\psi(1+v)+1/v+\sqrt{-1}\cot(\pi v)\Big(\psi(s-m/2)-\psi(s+m/2)\Big)\Bigg],$$

where $\beta_{j\mu}$ *(resp.* ξ_j*) are the eigenvalues of* $\chi(\alpha_\mu)$ *(resp.* $\chi\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)$*) when they are simultaneously diagonalized and* α_μ *is equal to* $\pm\begin{pmatrix}1 & v(\alpha_\mu)\\ 0 & 1\end{pmatrix}$. *Here we denote by* $\psi(s)$ *the logarithmic derivative of the gamma function.*

Starting from the results of [2], the proofs of these lemmas can be done by straight forward calculations.

Next we try to do the analytic continuation of $\Xi(s)$, $\Xi_{Eis}(s)$. For $\Xi(s)$, we must say few words. The left hand side of the Selberg trace formula for $T(\Gamma\alpha\Gamma)$ in our case has the form

$$\frac{\Xi'(s)}{\Xi(s)} = \sum_{\lambda_i} (2s-1)\ \mathrm{tr}(T(\Gamma\alpha\Gamma,\lambda_i))\left(\frac{1}{r_i^2+(s-1/2)^2} - \frac{1}{r_i^2+\beta^2}\right).$$

We can easily show that the terms $\mathrm{tr}(T(\Gamma\alpha\Gamma,\lambda_i))$ are uniformly bounded and the summation is absolutely convergent in the whole s-plane except for poles. So we can define $\Xi(s)$ up to some constant factor as a holomorphic function in $\mathrm{Re}(s)>1/2$. We note that $\Xi(s)$ is *not* meromorphic in whole s-plane because the numbers $\mathrm{tr}(T(\Gamma\alpha\Gamma,\lambda_i))$ are not necessarily integers.

For $\Xi_{Eis}(s)$, we have

$$\frac{\Xi'_{Eis}(s)}{\Xi_{Eis}(s)} = -\frac{1}{2s-1}\ \mathrm{tr}\ W(1/2)\Phi(1/2) + \frac{2s-1}{4\pi}\times$$

$$\int_{-\infty}^{\infty}\left(\frac{1}{r^2+(s-1/2)^2} - \frac{1}{r^2+\beta^2}\right)\mathrm{tr}(\ W(1/2+\sqrt{-1}r)\Phi'(1/2+\sqrt{-1}r)\Phi(1/2-\sqrt{-1}r))\ dr.$$

Put $\mathcal{F}(s) = \mathrm{tr}(W(s)\Phi'(s)\Phi(1-s))$. Then we have

$$\mathcal{F}(s) = \mathcal{F}(1-s)$$

by the general property $W(s)\Phi(s)=\Phi(s)W(1-s)$. By the functional equation of $\Phi(s)$, we know that all poles of $\mathcal{F}(s)$ are the poles of $\mathrm{tr}(\Phi'(s)\Phi(1-s))$. Recall that $W(s)$ is entire and bounded in any vertical strip. Thus each residue of the poles of $\mathcal{F}(s)$ is bounded. Let η be a pole of $\mathcal{F}(s)$ and A_η be its residue. Then we have

$$A_{\bar{\eta}} = \overline{A_\eta}$$

by $\Gamma\alpha\Gamma=\Gamma\alpha^{-1}\Gamma$. We can express $\mathcal{F}(s)$ by

$$\omega(s) + \sum_{\mathrm{Im}\,\eta\geq 0}\Big[\frac{\mathrm{Re}\,A_\eta}{s-\eta} + \frac{\mathrm{Re}\,A_\eta}{1-s-\bar{\eta}} + \sqrt{-1}\,\mathrm{Im}\,A_\eta\Big(\frac{1}{s-\eta} + \frac{1}{\eta} - \frac{1}{s-\bar{\eta}} - \frac{1}{\bar{\eta}}\Big)\Big],$$

where $\omega(s)$ is an entire function which satisfies $\omega(s)=\omega(1-s)$. Here, we must replace A_η by $A_\eta/2$ in the sum when $\mathrm{Im}\,\eta = 0$. The right hand side of this sum is absolutely convergent except η's. Using the Phragmén -Lindelöf principle, we have

$$\omega(s) = O(1)$$

in any vertical strip. Using the above expression of $\mathcal{F}(s)$, we can rewrite the right hand side of $\Xi'_{Eis}(s)/\Xi_{Eis}(s)$ in the form of a partial fraction

$$-\frac{1}{2s-1}\,\mathrm{tr}\,W(1/2)\Phi(1/2) + \frac{2s-1}{4\pi}\int_{-\infty}^{\infty}\frac{\omega(1/2+\sqrt{-1}r)}{r^2+(s-1/2)^2}\,dr + \sum_{\eta}\Big[\frac{\mathrm{Re}\,A_\eta}{1-s-\eta} + \frac{\mathrm{Re}\,A_\eta}{1-s-\bar{\eta}} + \sqrt{-1}\,\mathrm{Im}\,A_\eta\Big(\frac{1}{1-s-\eta} + \frac{1}{\eta} - \frac{1}{1-s-\bar{\eta}} - \frac{1}{\bar{\eta}}\Big)\Big],$$

where the last summation is taken over the poles η of $\mathcal{F}(s)$ which satisfy $\mathrm{Re}(\eta)>1/2$ and $\mathrm{Im}(\eta)\geq 0$. This summation is also absolutely convergent except for poles.

Finally for $\Xi_{hyp_{(1)}}(s)$, we have

$$\frac{\Xi'_{hyp_{(1)}}(s)}{\Xi_{hyp_{(1)}}(s)} = \sum_{[P]_{hyp_{(1)}}}\frac{(\mathrm{sgn}\,\mathrm{tr}P)^m\,\mathrm{tr}\,\chi(P)\,\log N(P_0)}{N(P)^{1/2} - N(P)^{-1/2}}\,N(P)^{-(s-1/2)},$$

where P_0 is a generator of the centralizer of P in $\Gamma/(\pm 1)$. The summation is absolutely convergent in $\mathrm{Re}(s)>1$. But now we found the analytic continuation of $\Xi'_{hyp_{(1)}}(s)/\Xi_{hyp_{(1)}}(s)$ by the analytic continuation of other terms of the Selberg trace formula.

§ 3. Proof of the theorem

Let $B = 2 + \sup_{\eta}(\mathrm{Re}(\eta))$, C_1 be the anticlockwise rectangular path which join $1-B-\sqrt{-1}T$, $B-\sqrt{-1}T$, $B+\sqrt{-1}T$, $1-B+\sqrt{-1}T$ and C_2 be the anticlockwise path which consists of three segments; from $1/2-\sqrt{-1}T$ to $B-\sqrt{-1}T$, from $B-\sqrt{-1}T$ to $B+\sqrt{-1}T$ and from $B+\sqrt{-1}T$ to $1/2+\sqrt{-1}T$. Without loss of generality, we assume there are no poles on C_1. Then we have

$$2\, N_{\Gamma\alpha\Gamma}(T) = \frac{1}{2\pi\sqrt{-1}} \int_{C_1} \Xi'(s)/\Xi(s)\, ds$$

$$= \frac{1}{\pi\sqrt{-1}} \int_{C_2} \Xi'(s)/\Xi(s)\, ds$$

by the functional equation $\Xi'(s)/\Xi(s) + \Xi'(1-s)/\Xi(1-s)=0$. By the arguments in § 2, we notice that each $\Xi_*(s)$ is a single valued meromorphic function in $\mathrm{Re}(s)>1/2$. So we can define the value on the line $\mathrm{Re}(s)=1/2$ by continuity. Thus

$$2\, N_{\Gamma\alpha\Gamma}(T) = \frac{1}{\pi}\Big[\arg \Xi_{ell}(s)\Xi_{hyp_{(1)}}(s)\Xi_{hyp_{(2)}}(s)\Xi_{par}(s)\Big]_{1/2-\sqrt{-1}T}^{1/2+\sqrt{-1}T}$$
$$+ \frac{1}{\pi}\int_{-T}^{T} \Xi_{Eis}'(1/2+\sqrt{-1}r)/\Xi_{Eis}(1/2+\sqrt{-1}r)dr + O(1),$$

because poles in $\mathrm{Re}(s)>1/2$ are only on the real axis. We note that

$$\Xi_{Eis}'(t)/\Xi_{Eis}(t) + \Xi_{Eis}'(1-t)/\Xi_{Eis}(1-t) = \mathcal{F}(t),$$

where t is sufficiently near the line $\mathrm{Re}(s)=1/2$. Using this fact, lemma 1~3 and the Stirling formula, we get

$$N_{\Gamma\alpha\Gamma}(T) - \frac{1}{4\pi} \int_{-T}^{T} \mathrm{tr}(\ W(1/2+\sqrt{-1}r)\Phi'(1/2+\sqrt{-1}r)\Phi(1/2-\sqrt{-1}r)\ )\ dr$$

$$= \frac{1}{\pi}\Big[\arg \Xi_{hyp_{(1)}}(s)\Big]_{1/2-\sqrt{-1}T}^{1/2+\sqrt{-1}T} + O(T \log T),$$

because

$$\arg \Xi_{ell}(1/2+\sqrt{-1}T)= O(\log T),$$
$$\arg \Xi_{hyp_{(2)}}(1/2+\sqrt{-1}T)= O(\log T),$$
$$\arg \Xi_{par}(1/2+\sqrt{-1}T)= O(T \log T).$$

Our final task is to estimate $\arg \Xi_{hyp_{(1)}}(1/2+\sqrt{-1}T)$. Using same type of argument as in Chapter 10, Theorem 2.24 of Hejhal [5], we have

$$\Xi'_{hyp_{(1)}}(s)/\Xi_{hyp_{(1)}}(s) = \sum_{|s-\rho|\le 1}\frac{T_\rho}{s-\rho} + \sum_{|s-\eta+1|\le 1}\frac{A_\eta}{s-\eta+1} + O(T),$$

where $T_\rho = tr(T(\Gamma\alpha\Gamma,\lambda))$. Noting that $|\log \Xi_{hyp_{(1)}}(s)|$ is sufficiently small when Re(s) is large, we see

$$\arg \Xi_{hyp_{(1)}}(B+\sqrt{-1}T) = O(1).$$

Recalling that T_ρ and A_η are uniformly bounded and $N_\Gamma(T)=O(T^2)$, we finally have

$$\arg \Xi_{hyp_{(1)}}(1/2+\sqrt{-1}T) = O(T).$$

This concludes the proof.

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