A self-similar tiling generated by the minimal Pisot number

Shigeki Akiyama Taizo Sadahiro

Abstract

Let β be a Pisot unit of degree 3 with a certain finiteness condition. A large family of self similar plane tilings can be constructed, by the digit expansion in base β. (cf. [7], [5], [8]) In this paper, we prove that the origin is an inner point of the central tile $\mathcal{K}$. Further, in the case corresponds to the minimal Pisot number, we shall give a detailed study on the fractal boundary of each tile. Namely, a sufficient condition of "adjacency" of tiles is given and the "vertex" of a tile is determined. Finally, we prove that the boundary of each tile is a union of 5 self similar sets of Hausdorff dimension $1.10026\ldots$.

1991 Mathematics Classification. Primary 11A68, 11R06
Key words and phrases. Fractal, Plane Tiling, Pisot number.

1 Plane tiling and Pisot numeration system

Let $\beta > 1$ be a real number. A *representation* in base β (or a β-representation) of a real number $x \geq 0$ is an infinite sequence $(x_i)_{k \geq i > -\infty}$, $x_i \geq 0$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain integer $k \geq 0$. It is denoted by

$$x = x_k x_{k-1} \cdots x_1 x_0 . x_{-1} x_{-2} \cdots .$$

A particular β-representation – called the β-expansion – can be computed by the 'greedy algorithm': Denote by $[y]$ and $\{y\}$ the integer part and the fractional part of y. There exists $k \in \mathbf{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = [x/\beta^k]$, and $r_k = \{x/\beta^k\}$. Then for $k > i > -\infty$, put $x_i = [\beta r_{i+1}]$, and $r_i = \{\beta r_{i+1}\}$. We get an expansion $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $k < 0$ ($x < 1$), we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion ends in infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted.

The digits x_i obtained by this algorithm are integers from the set $\mathcal{A} = \{0, \ldots, \beta - 1\}$ if β is an integer, or the set $\mathcal{A} = \{0, \ldots, [\beta]\}$ if β is not an integer.

A particular β-representation of 1, $d(1, \beta)$ is defined by means of the *β-transformation* of the unit interval:

$$T_\beta x = \{\beta x\}, \quad x \in [0, 1],$$
$$d(1, \beta) = 0.t_{-1}t_{-2}\ldots, \quad t_{-k} = [\beta T_\beta^{k-1} 1].$$

If a real number x has finite β-expansion, $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \ldots x_t\beta^t$, $(x_k, x_t \neq 0 \quad k \geq t)$, then we denote $\deg_\beta(x) = k$ and $\mathrm{ord}_\beta(x) = t$. Let $\mathbf{Fin}(\beta)$ be the set of numbers which have finite β-expansions. A *Pisot number* is an algebraic integer such that all its Galois conjugates other than itself are strictly inside the unit circle. A *Pisot numeration system* is a system to represent real numbers by the above greedy algorithm in a Pisot number base. In the following of this paper, we assume that β is a Pisot unit of degree 3 which is not totally real. (Here a Pisot unit means a Pisot number which is also a unit.)

We also assume that β has a property

$$\text{(F)} \qquad \mathbf{Z}[\beta]_{\geq 0} = \mathbf{Fin}(\beta).$$

A neccesary and sufficient condition of (F) for an arbitrary Pisot unit is shown in Theorem 2 of [1]. Here, we restate it, in our case.

Let β be the Pisot unit of degree 3 which is not totally real. Then β has the property (F) *if and only if each element of*

$$\{x \in \mathbf{Z}[\beta] \mid x \in [0,1], |x'| \le M_1\}$$

has finite β-expansion. Here M_1 is an upper bound of modulus of conjugates of all β-expansions $\sum_{i\ge 0} a_i\beta^i$.

Let $x_M x_{M-1} \cdots x_0 . x_{-1} \cdots x_{-N}$ be a sequence of integers in $\{0, 1, \ldots, [\beta]\}$ which includes the decimal point. Then $\mathbf{S}_{x_M x_{M-1} \cdots . \cdots x_{-N}}$ denotes the set of the numbers of the form

$$y_h\beta^h + y_{h-1}\beta^{h-1} + \cdots + x_M\beta^M + \cdots + x_{-N}\beta^{-N}, \quad (h \ge M)$$

which is a β-expansion. So if $x_M\beta^M + x_{M-1}\beta^{M-1} + \cdots + x_{-N}\beta^{-N}$ is not a β-expansion then $\mathbf{S}_{x_M \cdots . \cdots x_{-N}}$ is an empty set. For convenience we denote by

$$\mathcal{S} = \mathbf{S}_{.} = \{x \in \mathbf{Fin}(\beta) \mid \mathrm{ord}_\beta(x) \ge 0\}.$$

Let β' be a complex conjugate of β, and ϕ be the conjugate map which transforms β to β'. We denote $\phi(x)$ by x' and for any set $\mathbf{A} \subset \mathbf{Q}(\beta)$, $\mathbf{A}'$ denotes $\phi(\mathbf{A})$. It is clear that

$$\beta^{-1}\mathcal{S} = \mathcal{S} \cup \mathbf{S}_{.1} \cup \cdots \cup \mathbf{S}_{.[\beta]} \quad \text{(disjoint)}.$$

Conjugating both sides,

$$\beta'^{-1}\mathcal{S}' = \mathcal{S}' \cup \mathbf{S}'_{.1} \cup \cdots \cup \mathbf{S}'_{.[\beta]} \quad \text{(disjoint)}.$$

In general,

$$\mathbf{S}_a \cap \mathbf{S}_b \ne \emptyset \implies \mathbf{S}_a \subset \mathbf{S}_b \text{ or } \mathbf{S}_b \subset \mathbf{S}_a.$$

Let $K_{.x_{-1}\cdots x_{-N}}$ denote the closure of $\mathbf{S}'_{.x_{-1}\cdots x_{-N}}$ for the standard topology of $\mathbf{C}$ and $\mathcal{K}$ denote the closure of $\mathcal{S}'$. We say a set $K_{.x_{-1}x_{-2}\cdots x_{-M}}$ to be a *tile*.

This type of tiling constructed by the Pisot numeration system can be found in [7], [8] and [5]. We are interested in studying the topological structure of the tile, especially the structure of the boundary.

In this paper, we will prove that the origin is an inner point of $\mathcal{K}$ in Theorem 1, using geometry of numbers. This result is very much fundamental and give us good insight to understand Theorem 1 of [1].

In §3, we concentrate on the concrete Pisot number β, which is a positive root of $x^3 - x - 1$ and study the structure of the boundary in detail. A sufficient condition of the adjacency of two tiles is shown in Theorem 2, by constructing infinite points on the boundary. Here we used the fact $d(1,\beta) = .10001$, which means there exist two ways to carry up the digits:

$$1 = .011 \qquad \text{and} \qquad 1 = .10001\ .$$

This fact brings us an interesting feature of this Pisot numeration system. By the result of [6] and $d(1,\beta) = .10001$, an abstract sum $\sum a_i\beta^i$ with $a_i \in \{0,1\}$ is obtained as a β-expansion then

$$a_i = 1 \Longrightarrow a_{i+1} = a_{i+2} = a_{i+3} = a_{i+4} = 0. \tag{1}$$

The converse is also true if we forbid the exceptional expansion $100001000010000\cdots = (10000)^\infty$ in the tail of the sum $\sum a_i\beta^i$.

In Theorem 3, we prove that the 'vertex' of $\mathcal{K}$ forms a set which consists of five elements. Finally, in Theorem 4, we show that the boundary of $\mathcal{K}$ consists of 5 self similar set of Hausdorff dimension $1.10026\ldots$. (See figure 4.)

To prove these theorems, we develop a new technique called 'encircling method', which itself is of interest. We also employ the result proved in claim 1 of [1], which asserts that, for any greedy expansion $\sum_{i=0}^{N} a_i \beta^i$ with $a_0 = 1$, we have

$$|1 + a_1\beta' + a_2(\beta')^2 + \cdots + a_N(\beta')^N| > c_1|\beta'| \tag{2}$$

with $c_1 = 0.5752415728\ldots$.

In contrast with the result of [5] on the Pisot number defined by $x^3 - x^2 - x - 1$, we used geometry of numbers and Pisot numeration system instead of finite automata. The first author believes that this way is promissing in understanding universal phenomena of these tilings.

2 Fundamantal properties of the tiling

We shall show fundamentals of the tilings generated by cubic Pisot unit with (F), which will be needed later.

Lemma 1. *Let ξ be an arbitary point in $\mathbf{C}$. Then only a finite number of tiles can contain ξ, i.e.*

$$\#\{K_{.x_{-1}x_{-2}\cdots x_{-M}} \mid \xi \in K_{.x_{-1}x_{-2}\cdots x_{-M}}\} < +\infty$$

Proof. First, we show the set $C = \{x' \mid x \in \mathbf{Fin}(\beta),\ 0 < x < 1\}$ is a discrete set in $\mathbf{C}$. Suppose there is an accumulation point of C in $\mathbf{C}$, then 0 is also an accumulation point. For if there was a sequence (a'_n) which converges to z, then the sequence b'_n associated to $b_n = |a_{n+1} - a_n|$ belongs to C since $b_n > 0$ and $b_n \in \mathbf{Fin}(\beta)$. Moreover b'_n converges to 0. However for any $\epsilon > 0$, only finitely many algebraic integers can satisfy the condition, $|x| < 1, |x'| < \epsilon, |\overline{x'}| < \epsilon$. That is a contradiction. Assume that there exists infinite tiles $(K_i)\quad i = 1, 2, \cdots$ who include ξ. As each tile K_i must contain an element of C, and each tile has bounded diameter, there exist an accumulation point of C in $\mathbf{C}$. This proves the lemma. □

Lemma 2. *Let β be a Pisot number of degree 3 with property* (F)*, which is not totally real. Then* $\mathbf{C} = \bigcup_{.a_{-1}a_{-2}\cdots a_{-M}} K_{.a_{-1}a_{-2}\cdots a_{-M}}$.

Proof. By the property (F), it is clear that

$$\mathbf{Z}[\beta]_{>0} = \bigcup_{.a_{-1}a_{-2}\cdots a_{-M}} \mathbf{S}_{.a_{-1}a_{-2}\cdots a_{-M}},$$

where the summation is taken over all greedy expansions in base β of degree smaller or equal to 0. Thus it suffice to prove that the conjugate image of $\mathbf{Z}[\beta]_{>0}$ is dense in $\mathbf{C}$. Let

$$A = \begin{pmatrix} \Re\beta' & \Re\beta'^2 \\ \Im\beta' & \Im\beta'^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{|\beta'|^2}\begin{pmatrix} 2\Re\beta' \\ -1 \end{pmatrix}.$$

As $\{1, \beta, \beta^2\}$ is a base of $\mathbf{Q}(\beta)/\mathbf{Q}$, one can show that $1, v_1, v_2$ are linearly independent over $\mathbf{Q}$. Kronecker's approximation theorem implies the set $\mathbf{Z}$ (mod $\beta'\mathbf{Z}+\beta'^2\mathbf{Z}$) is dense in $\mathbf{C}/(\beta'\mathbf{Z}+\beta'^2\mathbf{Z})$. Let x be an arbitrary point of $\mathbf{C}$. For any $\varepsilon > 0$, there exist $(a, b, c) \in \mathbf{Z}^3$ such that $|x - a - b\beta' - c\beta'^2| \le \varepsilon/2$. Take a sufficiently large $M \in \mathbf{Z}$ such that $|\beta'^M| \le \varepsilon/2$ and $a + b\beta + c\beta^2 + \beta^M > 0$. Then $y = a + b\beta + c\beta^2 + \beta^M \in \mathbf{Z}[\beta]_{>0}$ satisfies $|x - y'| \le \varepsilon$. This proves the lemma.

□

Lemma 3. *Let $\beta > 1$ be an algebraic integer of degree k, which is a unit, and M be a positive number. Put*

$$X(p) = \{x \in \mathbf{Fin}(\beta) \mid |x| \le M, \mathrm{ord}_\beta(x) = -p\}.$$

Then we have

$$\lim_{p\to\infty} \min_{x\in X(p)} \max_{j=1,\dots,k-1} \left| x^{(j)} \right| = \infty.$$

Here $x^{(j)} \quad (j = 0, 1, 2, \dots k-1)$ *are all the conjugates of* x *with* $x^{(0)} = x$.

Proof. Denying the assertion and assume that there exist a constant B and an infinite sequence $x_i \quad (i = 1, 2, \dots)$ so that both

$$|x_i^{(j)}| \le B \quad \text{for } j = 1, 2, \dots k-1 \quad \text{and} \quad \lim_{i\to\infty} \mathrm{ord}_\beta(x_i) = -\infty$$

hold. Since β is a unit and $|x_i| \le M$, these x_i's are finite, a contradiction with the second condition. This shows the assertion. □

Here we can prove one of the main results of this paper.

Theorem 1. *Let* β *be a Pisot unit of degree* 3*, which is not totally real, with property* (F)*. Then for each element* $x \in \mathbf{Fin}(\beta)$ *with* $\mathrm{ord}_\beta(x) \ge 0$*,* x' *is an inner point of* $\mathcal{K}$.

Proof. First we show that 0 is an inner point of $\mathcal{K}$. By Lemma 1, for any positive B, there exists N such that $\mathrm{ord}_\beta(x) \le -N$ implies $|x'| > B$. (In fact, let $x = x_1 + x_2$ with $\mathrm{ord}_\beta(x_1) > 0, \deg_\beta(x_2) \le 0$. Then there exist an absolute constant C that $|x_1'| < C$. By using $\mathrm{ord}_\beta(x) = \mathrm{ord}_\beta(x_2) = -N$, we see for any positive B, there exist N such that $|x_2'| > B + C$, which shows $|x'| > B$.) Let $x = \beta^{-N+1}y$, then we have

$$\mathrm{ord}_\beta(y) < 0 \Longrightarrow |y'| > |\beta'|^{N-1}B.$$

Thus there are no elements of $\mathbf{Fin}(\beta)$ of negative order in the disk

$$U = \{\xi \in \mathbf{C} \quad | \quad |\xi| \le |\beta'|^{N-1}B\}.$$

By lemma 2, this shows that $U \subset \mathcal{K}$. Thus 0 is an inner point of $\mathcal{K}$. Let x be an element of $\mathbf{Fin}(\beta)$ whose expansion is

$$x = x_k x_{k-1} \cdots x_0.$$

Let m be the minimal length that $[\beta]0^{m-1}1 = [\beta]00\cdots01$ is lexicographically smaller than $d(1,\beta)$. Subdivide $\mathcal{K}$ in a form:

$$\mathcal{K} = \bigcup K_{x_{k+m}x_{k+m-1}\cdots x_0.},$$

where the index runs over all greedy expansion of length $m+k+1$. We see that 0 is also an inner point of $K_{0^{k+m+1}}$. By adding x, $\phi(x_k x_{k-1} \cdots x_0)$ is an inner point of $K_{0^m x_k x_{k-1} \cdots x_0}$. Thus the theorem is proved.

□

3 Tiling generated by the minimal Pisot number

In [5], [7], there exist extensive study on the tiling generated by the Pisot number of irreducible polynomial $x^3 - x^2 - x - 1$. This is now called the Rauzy fractal. It seems that few examples exist on other Pisot numbers. Hereafter in this paper, we concentrate on the minimal Pisot number β, which is the positive root of $x^3 - x - 1$. See Theorem 3.5 of [2]. Figure 3 shows $\mathcal{K}$ and $K_{.1}$ in this case. We believe that this example is interesting among others, because there exist two different ways to carry up:

$$1 = \beta^{-2} + \beta^{-3} \text{ and } 1 = \beta^{-1} + \beta^{-5}.$$

See also §9 and §10 of [8].

Propositon 1. *Let* β *be the real root of* $x^3 - x - 1 = 0$*. For any distinct tiles* T_1, T_2*,* $\mu(T_1 \cap T_2) = 0$*, where* μ *denotes Lebesgue measure.*

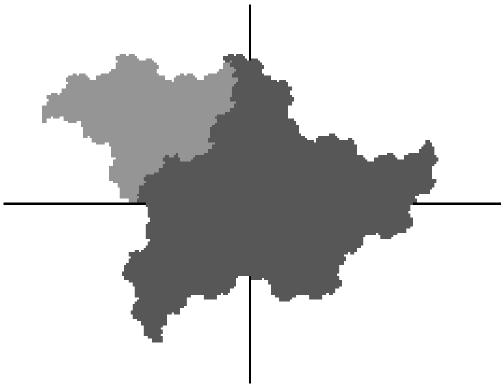

Figure 1: $\mathcal{K}$, $K_{.1}$ $: x^3 - x - 1$.

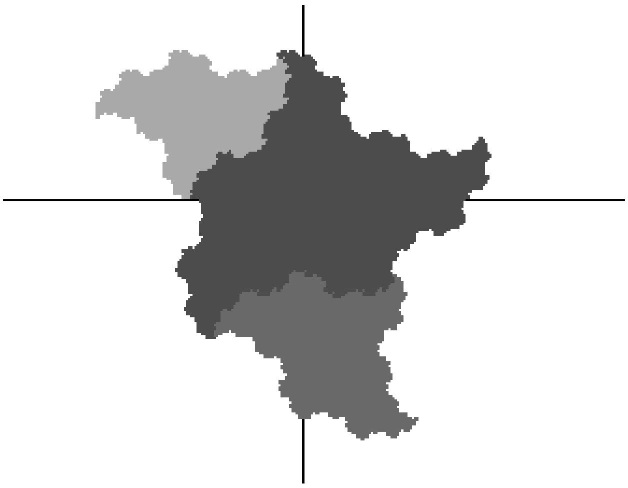

Figure 2: $\mathcal{K}$, $K_{.1}$, $K_{.01}$ $: x^3 - x - 1$.

Proof. It suffices to show when $T_1 = \mathcal{K}$, $T_2 = K_{.1}$. Recalling the admissiblity condition (1), $\mathcal{K} \cup K_{.1} = {\beta'}^{-1}\mathcal{K}$, $K_{.1} = {\beta'}^{-1} + {\beta'}^4\mathcal{K}$. So we have

$$\mu(\mathcal{K} \cup K_{.1}) = |{\beta'}^{-1}|^2\mu(\mathcal{K}) = \beta\mu(\mathcal{K}) \quad \mu(K_{.1}) = |{\beta'}^4|^2\mu(\mathcal{K}) = \beta^{-4}\mu(\mathcal{K}).$$

Thus

$$\begin{aligned}\mu(\mathcal{K} \cap K_{.1}) &= \mu(\mathcal{K}) + \mu(K_{.1}) - \mu(\mathcal{K} \cup K_{.1}) \\ &= (1 + \beta^{-4} - \beta)\mu(\mathcal{K}) \\ &= 0.\end{aligned}$$

□

Lemma 4. *Let β be the positive root of $x^3 - x - 1$. Then $\mathbf{Z}[\beta]_{\geq 0} = \mathbf{Fin}(\beta)$.*

Proof. We use theorem 2 in [1], recalled in §1. By using (1), we can take $M_1 = (1 - |\beta'|^5)^{-1}$. Let

$$C = \left\{ x \in \mathbf{Z}[\beta] \;\middle|\; 0 < x < 1, |x'| < \frac{1}{1 - |\beta'|^5} \right\}$$

Each element x of $\mathbf{Z}[\beta]$ can be written as $a+b\beta+c\beta^2$ with $a, b, c \in \mathbf{Z}$. Thus, for $x = a+b\beta+c\beta^2 \in C$,

$$\begin{pmatrix} 1 & \beta & \beta^2 \\ 1 & \beta' & {\beta'}^2 \\ 1 & \overline{\beta'} & \overline{{\beta'}^2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

with $0 \leq u < 1, |v| \leq 1/(1 - |\beta'|^5), |w| \leq 1/(1 - |\beta'|^5)$. Multiplying the inverse matrix and considering the absolute value, we see

$$|a| \leq 2, \qquad |b| \leq 1, \qquad |c| \leq 1.$$

By checking every element,

$$C = \{1, -1 + \beta, -1 + \beta^2, -\beta + \beta^2, 1 + \beta - \beta^2\}.$$

Now it is easy to show that these five elements have finite greedy expansion in base β. □

Lemma 5. *Let β be the real root of the polynomial $x^3 - x - 1$, and $(a_n) \subset \mathbf{Fin}(\beta)$ be a sequence of nonnegative numbers. Then $\lim_{n\to\infty} |a'_n| = 0$ if and only if $\lim_{n\to\infty} \mathrm{ord}_\beta(a_n) = \infty$.*

Proof. Suppose $h_n = \deg_\beta(a_n)$, $\ t_n = \mathrm{ord}_\beta(a_n)$ and

$$a_n = c_{n,h_n}\beta^{h_n} + c_{n,h_n-1}\beta^{h_n-1} + \cdots + c_{n,t_n}\beta^{t_n}.$$

Then

$$\begin{aligned}|a'_n| &\leq c_{n,h_n}|\beta'|^{h_n} + c_{n,h_n-1}|\beta'|^{h_n-1} + \cdots + c_{n,t_n}|\beta'|^{t_n} \\ &\leq [\beta]\frac{|\beta'|^{t_n}}{1 - |\beta'|} = [\beta]\frac{|\beta'|^{\mathrm{ord}_\beta(a_n)}}{1 - |\beta'|}.\end{aligned}$$

This shows $\mathrm{ord}_\beta(a_n) \to +\infty$ implies $|a'_n| \to 0$.

Assume there exist a sequence $(a_n)_{n>0}$, such that $\lim_{n\to+\infty} |a'_n| \to 0$ and $\mathrm{ord}_\beta(a_n) < B$. Then the sequence $b_n = a_n\beta^{-p-\mathrm{ord}_\beta(a_n)}$ satisfies $\lim_{n\to+\infty} |b'_n| \to 0$ and $\mathrm{ord}_\beta(b_n) = -p$. This contradicts the lemma 3.

□

Lemma 6. *If there exist two sequences $(a_n) \subset \mathbf{S}_{.x_{-1}\cdots x_{-N}}$, $(b_n) \subset \mathbf{S}_{.y_{-1}\cdots y_{-M}}$ such that $\{a'_n\}, \{b'_n\}$ converge in $\mathbf{C}$ and $\mathrm{ord}_\beta(|a_n - b_n|) \to \infty$, then $K_{.x_{-1}\cdots x_{-N}} \cap K_{.y_{-1}\cdots y_{-M}} \neq \emptyset$*

Proof. It follows from lemma 4 and 5. □

For example we can find a point of $K_{.} \cap K_{.1}$ by using $\beta^5 - 1 = \beta^4$ and $\beta^3 - 1 = \beta$.

$$\begin{array}{rcl} a_0 = 0 & b_0 = 0.1 = \beta^{-1} & |a_0 - b_0| = \beta^{-1} \\ a_1 = \beta^4 + a_0 & b_1 = b_0 & |a_1 - b_1| = |\beta^4 - \beta^{-1}| = \beta^3 \\ a_2 = a_1 & b_2 = b_1 + \beta^8 & |a_2 - b_2| = |\beta^8 - \beta^3| = \beta^7 \\ a_3 = \beta^{12} + a_2 & b_3 = b_2 & |a_3 - b_3| = |\beta^{12} - \beta^7| = \beta^{11} \\ \cdots & \cdots & \cdots \end{array}$$

In fact $\beta'^4 + \beta^{12} + \beta'^{20} + \cdots$ and $\beta'^{-1} + \beta'^8 + \beta'^{16} + \cdots$ converge to the same point $\beta'^4/(1-\beta'^8)$, which is in $K_{.} \cap K_{.1}$. Many other choices of $\{a_i\}$ and $\{b_i\}$ will produce other points of $K_{.} \cap K_{.1}$. We describe this construction more precisely as the following theorem.

Theorem 2. *Let β be the real root of the polynomial $x^3 - x - 1$. Let $a = 0.a_{-1} \cdots a_{-m}$ and $b = 0.b_{-1} \cdots b_{-n}$ be β-expansions of two nonnegative reals less than 1. If the following conditions are satisfied, then $K_{.a_{-1} \cdots a_{-m}} \cap K_{.b_{-1} \cdots b_{-n}}$ is an infinite set.*

- *For an integer $d \geq -5$, $a - b = \beta^d$.*
- $\deg_\beta(a) \leq d + 4, \quad \deg_\beta(b) \leq d$.

Proof. We define sequences (A_n) and (B_n) by the following procedure. Let $A_0 = a, B_0 = b$. When $A_n - B_n = \beta^{d_n}$, let $A_{n+1} = A_n$ and $B_{n+1} = B_n + \beta^{d_n + c_n}$ where $c_n = 5$ or 3. When $B_n - A_n = \beta^{d_n}$, let $A_{n+1} = A_n + \beta^{d_n + c_n}$ and $B_{n+1} = B_n$ where $c_n = 5$ or 3. If we choose $c_n = 5$ then

$$\begin{aligned} |A_{n+1} - B_{n+1}| &= \beta^{d_n+5} - |A_n - B_n| \\ &= \beta^{d_n+4}, \end{aligned}$$

and hence

$$\mathrm{ord}_\beta(|A_{n+1} - B_{n+1}|) = d_n + 4 = \mathrm{ord}_\beta(|A_n - B_n|) + 4.$$

Similarly, if we choose $c_n = 3$, then

$$\mathrm{ord}_\beta(|A_{n+1} - B_{n+1}|) = d_n + 1 = \mathrm{ord}_\beta(|A_n - B_n|) + 1.$$

Repeating this procedure, we can obtain sequences (A_n) and (B_n) such that $\mathrm{ord}_\beta(|A_n - B_n|) \to \infty$. But the condition $A_{n+1} \in \mathbf{S}_{.a_{-1} \cdots a_{-m}}$ and $B_{n+1} \in \mathbf{S}_{.b_{-1} \cdots b_{-n}}$ may not hold in the process. For example, let $A_0 = 0.1$ and $B_0 = 0.01$ then $A_0 - B_0 = \beta^{-6}$. If we put $A_1 = A_0$ and $B_1 = B_0 + \beta^{-6+c_0}$ where $c_0 = 3$ or 5 then $B_1 \notin \mathbf{S}_{.01}$ as it violates the condition (1). So we add the following restriction in choosing c_n. Let $M_n = \deg_\beta(\max\{A_n, B_n\}) - d_n, \quad m_n = \deg_\beta(\min\{A_n, B_n\}) - d_n$.

$$c_n = \begin{cases} 5 \text{ or } 3 & m_n \leq -2, d_n \geq -3 \\ 5 & m_n \leq -2, \ -4 \geq d_n \geq -5 \\ 5 & -2 < m_n \leq 0 \\ \text{stop} & \text{other} \end{cases}$$

Then $c_n - m_n \geq 5$ and $d_n + c_n \geq 0$. So by (1) in section 1, $A_{n+1} \in \mathbf{S}_{.a_{-1} \cdots a_{-m}}$ and $B_{n+1} \in \mathbf{S}_{.b_{-1} \cdots b_{-n}}$. If we can choose $c_n = 5$ then $(M_{n+1}, m_{n+1}) = (1, M_n - 4)$, and if we can choose $c_n = 3$ then $(M_{n+1}, m_{n+1}) = (2, M_n - 1)$.

We can take $A_0 = a, \quad B_0 = b, \ A_1 = a, \ B_1 = b + \beta^{d+5}$ and $A_2 = a + \beta^{d+9}, \ B_2 = b + \beta^{d+5}$ by the conditions of the theorem. For $d + 5 \geq \deg_\beta(b) + 5$, $d + 9 \geq \deg_\beta(a) + 5$ and $d + 5 \geq 0$.

Then $(M_2, m_2) = (1, -3)$ and $d_2 \geq -3$. Figure 3 shows the transition of (M_n, m_n) which starts from $(M_2, m_2) = (1, -3)$: An edge from (M, m) to (M', m') labeled by c means that, for $(M_n, m_n) = (M, m)$, if we choose $c_n = c$ then $(M_{n+1}, m_{n+1}) = (M', m')$. Distinct words $c_2 c_3 \ldots$ correspond to distinct sequences (A_n), (B_n). The set of right infinite words $c_2 c_3 \ldots$ obtained from this figure is $\{c_2 c_3 \cdots \mid c_i \in \{3, 5\}, c_i c_{i+1} \neq 33\}$ and hence is an infinite set. So we can find

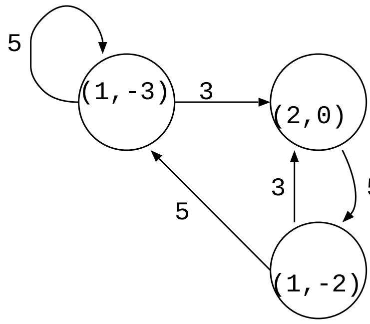

Figure 3: (M_n, m_n) and c_n

infinitely many sequences (A_n) of $\mathbf{S}_{.a_{-1}\ldots a_{-m}}$ and (B_n) of $\mathbf{S}_{.b_{-1}\ldots b_{-n}}$ which converge to points of $K_{.a_{-1}\cdots a_{-m}} \cap K_{.b_{-1}\cdots b_{-n}}$.

Let E_a be the set of all of the sequences (A_n) which are obtained from the process above. Then E_a is an infinite set. We have to show the set of limit points of elements of E_a is also an infinite set. Let x be a point of $\mathbf{C}$. Suppose $(A_j^{(i)})\quad(i = 1, 2, \cdots, N)$ are distinct N elements of E_a with $\lim_j (A_j^{(i)})' = x$. Then there exist a positive integer M that, for a sufficiently large j, the fractional part of $\beta^{-M} A_j^{(i)}$ are distict. This means that $\beta'^{-M} x$ is contained in distinct N tiles. By lemma 1, only finitely many tiles can contain a same point, we see that N is bounded. Thus the set of limit points of elements of E_a is also an infinite set.

□

Figure 5 shows the subset of $\mathcal{K} \cap (K_{.1} \cup K_{.01} \cup K_{.001} \cup K_{.0001} \cup K_{.00001})$ obtained by the process above. The second auther predicts that these points are all of the intersection. We shall prove later that this boundary can also be obtained by iterated function system.

Lemma 7. *Consider five tiles $\mathcal{K}, K_{.0001}, K_{.1000001}, K_{.0000001}, K_{.01}$ in this order. We consider this order "cyclically". In other words, we consider that $\mathcal{K}$ and $K_{.01}$ are also adjacent. Then two adjacent tiles in this sense have infinitely many points in common. And two tiles, which are not adjacent in this sense, have only one point $-\beta'^{-3}$ in common. Moreover, $-\beta'^{-3}$ is an inner point of $\mathcal{K} \cup K_{.0001} \cup K_{.1000001} \cup K_{.0000001} \cup K_{.01}$.*

Roughly speaking, this lemma assures that these 5 tiles are "actually" adjacent in this order and surround a point $-\beta'^{-3}$. (See Figure 5,6.)

Proof. First we prove

$$\mathcal{K} \cap K_{.0000001} = \{-\beta'^{-3}\}. \tag{3}$$

Other equalities of this type are proved similarly. Consider two subdivisions:

$$\mathcal{K} = K_{0000.} \cup K_{1000.} \cup K_{100.} \cup K_{10.} \cup K_{1.}$$

and

$$K_{.0000001} = K_{0000.0000001} \cup K_{1000.0000001} \cup K_{100.0000001} \cup K_{10.0000001} \cup K_{1.0000001}.$$

Let $D(x, r) = \{z \in \mathbf{C} \mid |z - x| \leq r\}$. We have the inclusion (cf. Lemma 4), $\mathcal{K} \subset D(0, v)$ with $v = 1/(1 - |\beta'|^5)$. Thus $K_{1000.} = K_{00001000.} \subset D(\beta'^3, v|\beta'|^8|)$, etc. By using this, we see

$$K_{0000.} \cup K_{1000.} \cup K_{100.} \cup K_{1.} \subset D(0, v|\beta'|^4) \cup D(\beta'^3, v|\beta'|^8) \cup D(\beta'^2, v|\beta'^7|) \cup D(1, v|\beta'|^5)$$

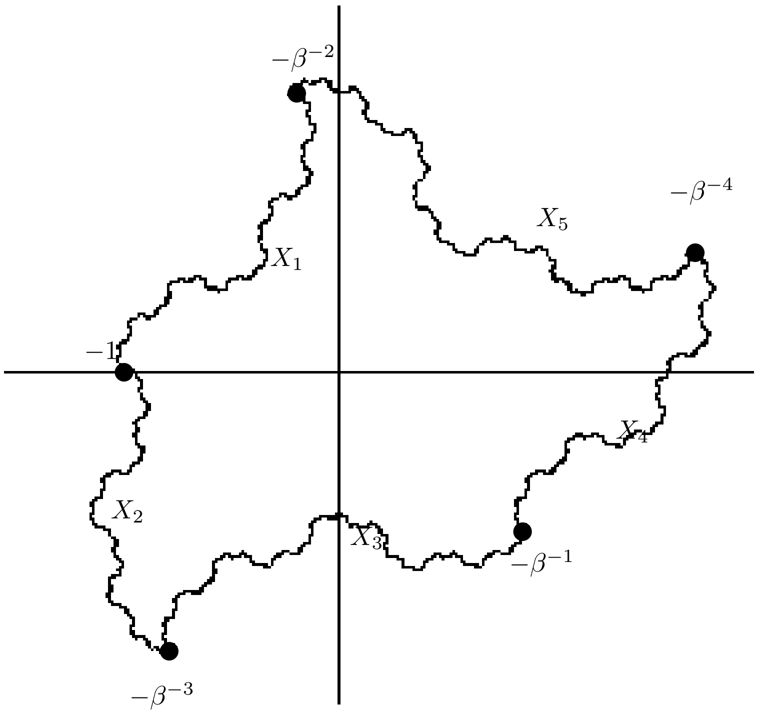

Figure 4: subset of $\mathcal{K} \cap (K_{.1} \cup K_{.01} \cup K_{.001} \cup K_{.0001} \cup K_{.00001})$.

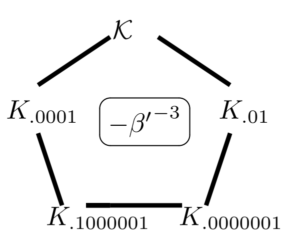

Figure 5: adjacency of $\mathcal{K}, K_{.0001}, K_{.1000001}, K_{.0000001}$ and $K_{.01}$.

and

$$
\begin{aligned}
K_{.0000001} \quad \subset \quad & D({\beta'}^{-7}, v|\beta'|^4) \cup D({\beta'}^3 + {\beta'}^{-7}, v|\beta'|^8) \cup \\
& D({\beta'}^2 + {\beta'}^{-7}, v|{\beta'}^7|) \cup D(\beta' + {\beta'}^{-7}, v|\beta'|^6) \cup D(1 + {\beta'}^{-7}, v|\beta'|^5).
\end{aligned}
$$

Hence

$$K_{.0000001} \cap (K_{0000.} \cup K_{1000.} \cup K_{100.} \cup K_{1.}) = \emptyset. \quad \text{(See Figure 6, 7.)} \tag{4}$$

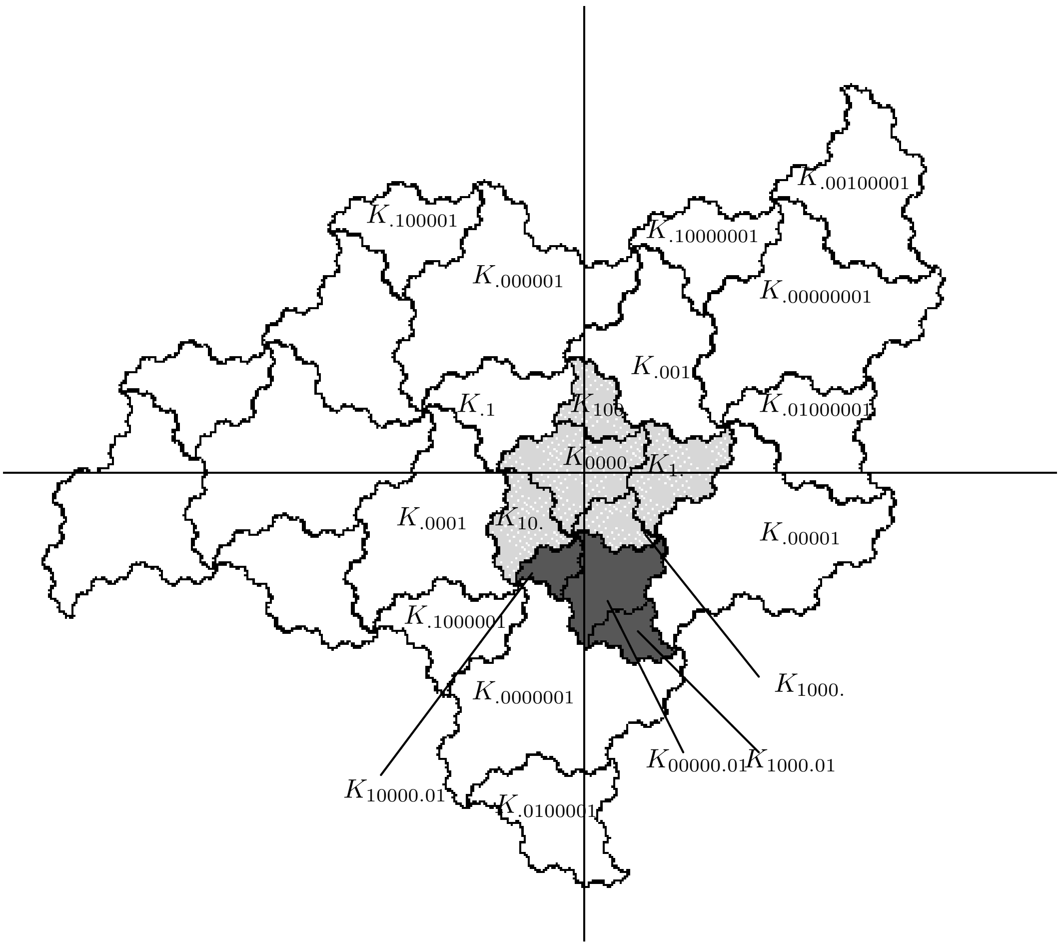

Figure 6: Subdivision of tiles

In the same manner

$$\mathcal{K} \cap (K_{0000.0000001} \cup K_{1000.0000001} \cup K_{10.0000001} \cup K_{1.0000001}) = \emptyset. \tag{5}$$

Hereafter we call this type of arguments "encircling method". Combining (4) and (5),

$$\mathcal{K} \cap K_{.0000001} = K_{10.} \cap K_{100.0000001} = K_{0.} \cap K_{00.0000001}. \tag{6}$$

Here we used

$$K_{0.} = K_{10.} \cup K_{0000.} \cup K_{100.} \cup K_{1000.}$$

and

$$K_{00.0000001} = K_{100.0000001} \cup K_{0000.0000001} \cup K_{1000.0000001}.$$

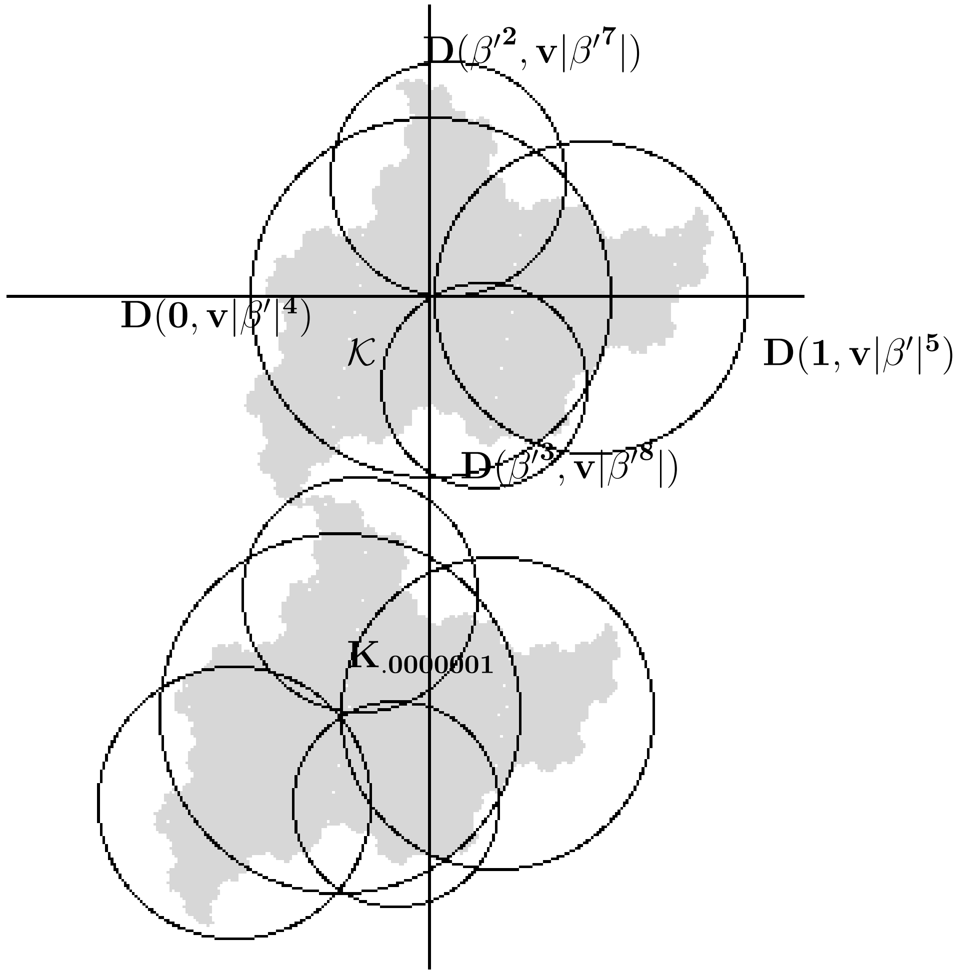

Figure 7: $K_{.0000001} \cap \Big(D(0, v|\beta'|^4) \cup D(\beta'^3, v|\beta'|^8) \cup D(\beta'^2, v|\beta'^7|) \cup D(1, v|\beta'|^5)\Big) = \emptyset$.

Now let $\psi_1'(z) = \beta'^5 z + \beta'$, then

$$\psi_1'(K_{0.}) = K_{10.} \tag{7}$$

and

$$\psi_1'(K_{00.0000001}) = K_{100.0000001} \tag{8}$$

hold. For the convenience of the reader, we will show (8) precisely. It suffice to show $\psi_1(\mathbf{S}_{00.0000001}) = \mathbf{S}_{100.000001}$ with $\psi_1(z) = \beta^5 z + \beta$. Take any $x = \cdots 00.0000001$. Then

$$\begin{aligned} \psi_1(x) &= \cdots 0000000.01 + 10. \\ &= \cdots 0000010.01 \\ &= \cdots 0000010.0010001 \\ &= \cdots 0000100.0000001. \end{aligned}$$

Here we used $\beta^5 = \beta^4 + 1$ twice. Note that, as there exist 4 consecutive 0's in front of 100.000001, $\psi_1(x)$ lies in $\mathbf{S}_{100.000001}$ by (1). This shows $\psi_1(\mathbf{S}_{00.0000001}) \subset \mathbf{S}_{100.000001}$. Similarly, $\psi^{-1}(\mathbf{S}_{100.000001}) \subset \mathbf{S}_{00.0000001}$. This shows (8). Hereafter we omit the details of such calculation in Pisot numeration system. By using (6), (7),(8), we have a set equation

$$K_{0.} \cap K_{00.0000001} = \psi_1'(K_{0.} \cap K_{00.0000001}).$$

This shows that $K_{0.} \cap K_{00.0000001}$ consists of all fixed points of ψ_1'. Therefore, $K_{0.} \cap K_{00.0000001} = \{-\beta'^{-3}\}$, which shows (3).

Now consider the map $\eta(z) = \beta' z + {\beta'}^{-7}$. Then one can show that

$$K_{0.} \xrightarrow{\eta} K_{00.0000001} \xrightarrow{\eta} K_{000.0001} \xrightarrow{\eta} K_{0000.01} \xrightarrow{\eta} K_{0000.10000001} \xrightarrow{\eta} K_{0.}$$

By Lemma 6, we can show that $\{-1, -{\beta'}^{-3}\} \subset K_{0.} \cap K_{000.0001}$. As η is a contraction map with fixed point $-{\beta'}^{-3}$, $\{\eta^{5n}(-1) \mid n = 0, 1, \cdots\}$ is an infinite set contained in $K_{0.} \cap K_{000.0001} \subset \mathcal{K} \cap K_{.0001}$. By considering $\{\eta^{5n+k}(-1) \mid n = 0, 1, \cdots\}$ for $k = 1, 2, 3, 4$, adjacent tiles in the above sense have infinitely many common points.

It remains to prove that $-{\beta'}^{-3}$ is an inner point of $\mathcal{K} \cup K_{.0001} \cup K_{.1000001} \cup K_{.0000001} \cup K_{.01}$. One can show that $-{\beta'}^{4}$ is an inner point of $\mathcal{K}$. In fact, by using (2), $|{\beta'}^4| < 0.570 < c_1$. This implies that $-{\beta'}^{-3}$ is an inner point of ${\beta'}^{-7}\mathcal{K}$, which is subdivided into 11 tiles:

$$\begin{aligned} {\beta'}^{-7}\mathcal{K} \quad = \quad & \mathcal{K} \cup K_{.1} \cup K_{.01} \cup K_{.001} \cup K_{.0001} \cup \\ & K_{.00001} \cup K_{.000001} \cup K_{.100001} \cup \\ & K_{.0000001} \cup K_{.0100001} \cup K_{.1000001}, \end{aligned}$$

as there exist 11 different β-expansions of length 7. By using encircling method, one can easily show that 6 tiles:

$$K_{.1}, K_{.001}, K_{.00001}, K_{.000001}, K_{.100001}, K_{.0100001}$$

do not contain $-{\beta'}^{-3}$. This completes the proof. □

We call a set $K_{x_N x_{N-1} \cdots x_0 . x_{-1} x_{-2} \cdots x_{-M}}$ to be a *subtile* of the tile $K_{.x_{-1} x_{-2} \cdots x_{-M}}$. All subtiles are similar to $\mathcal{K}$. A common point of two tiles is called an element of the *boundary* of the tiling. A common point of three tiles is called an element of the *vertex* of the tiling. We define $\partial(K_{.x_{-1} x_{-2} \cdots x_{-M}})$ to be the set of all boundaries of a tile $K_{.x_{-1} x_{-2} \cdots x_{-M}}$ and $V(K_{.x_{-1} x_{-2} \cdots x_{-M}})$ to be the set of all vertices in $K_{.x_{-1} x_{-2} \cdots x_{-M}}$.

Theorem 3. *We have*

$$V(\mathcal{K}) = \{-1, -{\beta'}^{-1}, -{\beta'}^{-2}, -{\beta'}^{-3}, -{\beta'}^{-4}\}. \qquad \text{(See Figure4.)}$$

Proof. We use the same notation as in the proof of Lemma 7. Remembering $\mathcal{K} \subset D(0, v)$, and $D(0, c_1) \subset \mathcal{K}$. As $c_1 |\beta'|^{-9} > v$, each element of $\mathcal{K}$ is an inner point of ${\beta'}^{-9}\mathcal{K}$. Thus it is enough to consider ${\beta'}^{-9}\mathcal{K}$, which is subdivided into 20 tiles. (See Figure 6. Remark that there exist subtiles in this figure.) There are only 10 tiles which have at least one common point with $\mathcal{K}$, namely

$$K_{.1}, K_{.01}, K_{.001}, K_{.0001}, K_{.00001}, K_{.000001}, K_{.0000001}, K_{.1000001}, K_{.00000001}, K_{.01000001}.$$

(By using this, we see that all element of $\mathcal{K}$ is an inner point of ${\beta'}^{-8}\mathcal{K}$, actually.) Here we used a *refined* version of encircling method. For example, to prove $\mathcal{K} \cap K_{.000000001} = \emptyset$, we subdivide $\mathcal{K}$ into 5 subtiles: $K_{1.}, K_{10.}, K_{100.}, K_{1000.}, K_{0000.}$ and use the encircling method. Hereafter we will use this technique, if necessary. The numerical value of the radius $v = 1.9806$ can be replaced by 1.85 but we cannot expect 1.8, as

$$|1 + {\beta'}^5 + {\beta'}^{10} + {\beta'}^{15} + {\beta'}^{21} + {\beta'}^{26} + {\beta'}^{31}| = 1.80809\ldots.$$

However we cannot confirm $\mathcal{K} \cap K_{.000000001} = \emptyset$, by simple encircling method with radius 1.8. Thus we cannot get rid of this refined version. For the simplicity, we also call this refined version as "encircling method".

By using the results and the proof of Lemma 7, multiplying ${\beta'}^{-1}$, the same kind of statements holds for the 5 subtiles: $K_{1.}, K_{.00001}, K_{.01000001}, K_{.00000001}, K_{.001}$ surround the point $-{\beta'}^{-4}$ in this order. In the same way, multiplying β', we see that the 5 subtiles $K_{100.}, K_{.001}, K_{1.000001}, K_{0.000001}, K_{.1}$ surround $-{\beta'}^{-2}$ in this order. Noting

$$K_{.000001} = K_{1.000001} \cup K_{0.000001},$$

we should say that 4 tiles $K_{100.}, K_{.001}, K_{.000001}, K_{.1}$ surround $-\beta'^{-2}$ in this order and satisfy similar properties. Continuing this process, 4 subtiles: $K_{1000.}, K_{.01}, K_{.00001}, K_{1.}$ surround $-\beta'^{-1}$ in this order and satisfy similar properties. However, noting $K_{1000.} \cup K_{1.} \subset \mathcal{K}$, 3 tiles $\mathcal{K}, K_{.01}, K_{.00001}$ surround $-\beta'^{-1}$ and $\mathcal{K} \cap K_{.01} \cap K_{.00001} = \{-\beta'^{-1}\}$. (In fact, by using encircling method, $\mathcal{K} \setminus (K_{1000.} \cup K_{1.}) = K_{10.} \cup K_{0000.} \cup K_{100.}$ does not have any intersection with $K_{.00001}$.) Similarly, 3 tiles $\mathcal{K}, K_{.1}, K_{.0001}$ surround -1 and $\mathcal{K} \cap K_{.1} \cap K_{.0001} = \{-1\}$. Summarizing these facts, we have shown $V(\mathcal{K}) \supset \{-1, -\beta'^{-1}, -\beta'^{-2}, -\beta'^{-3}, -\beta'^{-4}\}$, and

$$\begin{aligned} K_{.000001} \cap \mathcal{K} &= \{-\beta'^{-2}\} \\ K_{.0000001} \cap \mathcal{K} &= \{-\beta'^{-3}\} \\ K_{.1000001} \cap \mathcal{K} &= \{-\beta'^{-3}\} \\ K_{.00000001} \cap \mathcal{K} &= \{-\beta'^{-4}\} \\ K_{.01000001} \cap \mathcal{K} &= \{-\beta'^{-4}\}. \end{aligned}$$

Thus, to prove $V(\mathcal{K}) \subset \{-1, -\beta'^{-1}, -\beta'^{-2}, -\beta'^{-3}, -\beta'^{-4}\}$, we only have to consider 5 tiles $K_{.1}, K_{.01}$, $K_{.001}$, $K_{.0001}, K_{.00001}$. Again, by using encircling method,

$$\begin{aligned} (K_{.1} \cup K_{.001}) \cap K_{.01} &= \emptyset \\ (K_{.0001} \cup K_{.1}) \cap K_{.00001} &= \emptyset \\ (K_{.001} \cup K_{.0001}) \cap K_{.001} &= \emptyset \\ (K_{.00001} \cup K_{.001}) \cap K_{.1} &= \emptyset \\ (K_{.001} \cup K_{.00001}) \cap K_{.0001} &= \emptyset. \end{aligned}$$

This proves the assertion. □

Theorem 4. *The boundary of* $\mathcal{K}$ *is a union of* 5 *self affine sets. More precisely, define two types of contraction maps* $\psi'_i(z) = \beta'^5 z + \beta'^i$ *and* $\phi'_j(z) = \beta'^4 z + \beta'^j$. *Then we have*

$$\partial(\mathcal{K}) = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5,$$

and each X_i *is defined by a set equation:*

$$\begin{aligned} X_1 &= \psi'_2(X_1) \cup \phi'_{-1}(X_1) \\ X_2 &= \psi'_1(X_2) \cup \phi'_{-4}(X_2) \\ X_3 &= \psi'_1(X_3) \cup \phi'_{-2}(X_3) \\ X_4 &= \psi'_0(X_4) \cup \phi'_{-5}(X_4) \\ X_5 &= \psi'_0(X_5) \cup \phi'_{-3}(X_5). \end{aligned}$$

We also have

$$\begin{aligned} X_1 \cap X_2 &= \{-1\} \\ X_2 \cap X_3 &= \{-\beta'^{-3}\} \\ X_3 \cap X_4 &= \{-\beta'^{-1}\} \\ X_4 \cap X_5 &= \{-\beta'^{-4}\} \\ X_5 \cap X_1 &= \{-\beta'^{-2}\}. \end{aligned}$$

The Hausdorff dimension of the boundary is $1.10026\ldots$.(See Figure 4.)

According to this theorem, we say that X_1 is an *edge* between $-\beta'^{-2}$ and -1, which is denoted by $E(-\beta'^{-2}, -1)$, and so on.

Proof. In the proof of Theorem 2, we already showed that

$$\partial(\mathcal{K}) = \mathcal{K} \cap (K_{.1} \cup K_{.0001} \cup K_{.01} \cup K_{.00001} \cup K_{.001}) .$$

Thus define

$$\begin{aligned} X_1 &= \mathcal{K} \cap K_{.1} \\ X_2 &= \mathcal{K} \cap K_{.0001} \\ X_3 &= \mathcal{K} \cap K_{.01} \\ X_4 &= \mathcal{K} \cap K_{.00001} \\ X_5 &= \mathcal{K} \cap K_{.001}. \end{aligned}$$

See Figure 4. First, we consider X_3. Note that

$$\mathcal{K} = K_{1.} \cup K_{10.} \cup K_{100.} \cup K_{1000.} \cup K_{10000.} \cup K_{00000.}$$

and

$$K_{.01} = K_{10000.01} \cup K_{00000.01} \cup K_{1000.01}.$$

By using encircling method, we have

$$(K_{100.} \cup K_{10000.}) \cap K_{.01} = \emptyset$$

and

$$K_{1000.01} \cap \mathcal{K} = \emptyset.$$

Thus

$$\mathcal{K} \cap K_{.01} = (K_{1.} \cup K_{10.} \cup K_{1000.} \cup K_{00000.}) \cap (K_{10000.01} \cup K_{00000.01}) .$$

By using Lemma 7 and multiplying ${\beta'}^5$, we see 5 subtiles $K_{00000.}, K_{10.}, K_{10000.01}, K_{00000.01}, K_{1000.}$ surround $-{\beta'}^2$ in this order and satisfy similar properties. Especially, $K_{00000.} \cap K_{0000.01} = \{-{\beta'}^2\}$. In the proof of Theorem 3, we already showed that $K_{1.} \cap K_{.01} = \{-{\beta'}^{-1}\}$. Summing up

$$\mathcal{K} \cap K_{.01} = (K_{10.} \cup K_{1000.}) \cap (K_{10000.01} \cup K_{00000.01}) = K_{0.} \cup K_{0000.01}.$$

Further, by using $K_{10.} \cap K_{00000.01} = \{-{\beta'}^2\}$ and $K_{1000.} \cap K_{10000.01} = \{-{\beta'}^2\}$,

$$X_3 = (K_{10.} \cap K_{10000.01}) \cup (K_{1000.} \cap K_{00000.01}) = K_{0.} \cup K_{0000.01}.$$

Now,

$$\begin{aligned} \psi'_1(K_{0.}) &= K_{10.} \\ \psi'_1(K_{0000.01}) &= K_{10000.01} \\ \phi'_{-2}(K_{0.}) &= K_{00000.01} \\ \phi'_{-2}(K_{0000.01}) &= K_{1000.}. \end{aligned}$$

Thus we have a set equation $X_3 = \psi'_1(X_3) \cup \phi'_{-2}(X_3)$. Other set equations can be shown similarly. But there exists a short cut. It is easy to show $X_1 = \beta' X_3$ and $X_5 = {\beta'}^{-1} X_3$. One see that X_2 is a subset of ${\beta'}^{-1} X_5$. To be precise, $X_2 = \phi'_4({\beta'}^{-1} X_5)$. For X_4, we can show $X_4 = {\beta'}^{-1} X_2$. These relations are enough to prove the set equations for X_1, X_2, X_4, X_5. The relations

$$\begin{aligned} X_1 \cap X_2 &= \{-1\} \\ X_2 \cap X_3 &= \{-{\beta'}^{-3}\} \\ X_3 \cap X_4 &= \{-{\beta'}^{-1}\} \\ X_4 \cap X_5 &= \{-{\beta'}^{-4}\} \\ X_5 \cap X_1 &= \{-{\beta'}^{-2}\}, \end{aligned}$$

are easily shown by similar arguments.

It remains to determine the Hausdorff dimension of $\partial(\mathcal{K})$. As all X_i is similar to X_3, we treat only X_3. Our obstacle is the fact that the iterated function system $\{\psi_1, \phi_{-2}\}$ seems not satisfy open set condition. But we use the criterion of Exercise 3.3 in [3], to show that the Hausdorff dimension s of X_3 coincides with the upper and lower box counting dimension and the Hausdorff measure $\mathcal{H}^s(X_3)$ is positive. It is also proved in Corollary 3.3 of [3], that $\mathcal{H}^s(X_3) < \infty$. Noting $\psi_1(X_3) \cap \phi_{-2}(X_3) = \{-\beta'^2\}$,

$$\mathcal{H}^s(X_3) = \mathcal{H}^s(\psi_1(X_3)) + \mathcal{H}^s(\phi_{-2}(X_3)) = |\beta'^{5s}|\mathcal{H}^s(X_3) + |\beta'^{4s}|\mathcal{H}^s(X_3).$$

Thus

$$1 = |\beta'|^{5s} + |\beta'|^{4s},$$

which shows that $s = 1.10026\cdots$.

$\square$

Acknowledgements

The authors really appreciate the kindness of the refree, who read the original manuscript carefully and gave us helpful comments.

References

[1] S.Akiyama *Pisot numbers and greedy algorithm.* Number Theory, Diophantine, Cumputational and Algebraic Aspects, edited by K.Györy, A. Pethö and V.T.Sós, 9-21, de Gruyter, Berlin, New York (1998)

[2] M.J.Bertin, A.Decomps-Guilloux, M.Grandet-Hugot, M.Pathiaux-Delefosse, J.P.Schreiber *Pisot and Salem Numbers.* Birkhäuser Verlag Basel·Boston ·Berlin. (1992)

[3] K.J.Falconer *Techniques in Fractal Geometry.* John Wiley & Sons.(1997)

[4] C.Frougny, B.Solomyak *Finite beta-expansions.* Ergod. Th. and Dynam. Sys. 12 (1992) 713-723

[5] A.Messaoudi *Frontière du fractal de Rauzy et système de numeration complexe.* preprint.

[6] W.Parry *On the β-expansions of real numbers.* Acta Math. Acad. Sci. Hungar. 11 (1960) 269-278

[7] G.Rauzy *Nombres Algébriques et substitutions.* Bull. Soc. France 110 (1982), 147-178.

[8] W.P.Thurston *Groups, Tilings and Finite state automata.* Summer 1989 AMS Colloquium lectures, Research Report GCG 1.

Shigeki Akiyama
Dept. of Math., Fac. of Science
Niigata University,
Ikarashi-2 8050, Niigata
950-2181, JAPAN
E-mail: akiyama@math.sc.niigata-u.ac.jp

Taizo Sadahiro
Central Computer Room
Prefectural University of Kumamoto
3-1-100 Tsukide, Kumamoto
862-8502 Japan
E-mail: sadahiro@pu-kumamoto.ac.jp