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Lehmer Numbers and an Asymptotic Formula for π

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Let $\{a_n\}_{n=0}^{\infty}$ be an integer sequence defined by the non-degenerate binary linear recurrence $a_n = A\,a_{n-1} + Ba_{n-2}$, where $a_0 = 0$, $a_1 \neq 0$, and A, B are fixed non-zero integers. It is proved, for a certain constant κ, that

$$\left(\frac{6(1-\kappa)\log|a_1 a_2 \cdots a_n|}{\log[a_1, a_2, ..., a_n]}\right)^{1/2} = \pi + O\left(\frac{1}{\log n}\right),$$

which is the generalization of the formula of P. Kiss and F. Mátyás.

Let $\{a_n\}_{n=0}^{\infty}$ be an integer sequence defined by the binary linear recurrence

$$a_n = Aa_{n-1} + Ba_{n-2},$$

where $a_0 = 0$, $a_1 \neq 0$, and A, B are fixed non-zero integers. Denote by α, β the roots of the characteristic polynomial $x^2 - Ax - B$. We assume that $|\alpha| \geqslant |\beta|$ and α/β is not a root of unity.

P. Kiss and F. Mátyás [1] showed, subject to the conditions $(A, B) = 1$ and $a_1 = 1$, that

$$\left(\frac{6\log|a_1 a_2 \cdots a_n|}{\log[a_1, a_2, ..., a_n]}\right)^{1/2} = \pi + O\left(\frac{1}{\log n}\right), \qquad (1)$$

where $[a_1, a_2, ..., a_n]$ denotes the least common multiple of the terms $a_1, a_2, ..., a_n$.

In this paper, we show

THEOREM. *Let $\kappa = \log(A^2, B)/2\log|\alpha|$. Then we have*

$$\left(\frac{6(1-\kappa)\log|a_1 a_2 \cdots a_n|}{\log[a_1, a_2, ..., a_n]}\right)^{1/2} = \pi + O\left(\frac{1}{\log n}\right).$$

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Remark. The formula (1) follows from our theorem because if $(A, B)=1$ then $\kappa=0$.

To prove our theorem, we first recall notations and some results about Lehmer numbers. Let γ, δ be the complex numbers satisfying

$$(\gamma+\delta)^2=C, \qquad \gamma\delta=D,$$

where C, D are coprime non-zero integers. We assume $|\gamma|\geqslant|\delta|$ and γ/δ is not a root of unity. Then the Lehmer numbers L_n associated with γ, δ are defined by

$$L_n=\begin{cases}(\gamma^n-\delta^n)/(\gamma-\delta), & \text{for } n \text{ odd},\\ (\gamma^n-\delta^n)/(\gamma^2-\delta^2), & \text{for } n \text{ even}.\end{cases}$$

Note that $\{L_n\}_{n=0}^{\infty}$ is an integer sequence. This sequence is introduced by Lehmer [3] and applied to the primality test of the numbers of type $A2^p-1$.

LEMMA 1.

$$\log\,[L_1, L_2, ..., L_n]=\frac{3\log|\gamma|}{\pi^2}n^2+O\left(\frac{n^2}{\log n}\right).$$

Proof. Let p be a prime and denote by $p^e \| L_n$ when $p^e | L_n$ and $p^{e+1} \nmid L_n$ for $e\geqslant 1$. Let T_n be the product of p^e's with $p^e \| L_n$ and $p \nmid L_1 L_2 \cdots L_{n-1}$. Then we have, for $n\geqslant 13$,

$$T_n=\lambda_n^{-1}\prod_{d|n}(\gamma^d-\delta^d)^{\mu(n/d)},$$

where $\mu(\cdot)$ is the Möbius function and λ_n is equal to 1 or the greatest prime divisor of $n/(3, n)$ (see [4, Lemmas 6, 7, 8; 2, Lemma 1]).

To prove (1), P. Kiss and F. Mátyás [1] used this expression of T_n essentially when C is a square, and showed the asymptotic formula of our Lemma 1 (see [1, Lemma 2, 3]). We can easily see that these arguments are also true when C is an arbitrary non-zero integer.

LEMMA 2.

$$\log|L_1 L_2 \cdots L_n|=\frac{\log|\gamma|}{2}n^2+O(n\log n).$$

Proof. In the light of Baker's method, we have

$$|1-\omega^n|>n^{-c},$$

where ω is an algebraic number whose absolute value is 1 and ω is not a root of unity. The constant c depends only on ω (see [2, Lemma 3]). Thus we have

$$|\gamma|^n n^{-c_1}\leqslant|\gamma^n-\delta^n|\leqslant c_2|\gamma|^n$$

and

$$|\gamma|^n n^{-c_3}\leqslant|L_n|\leqslant c_4|\gamma|^n,$$

where c_i $(i=1, 2, 3, 4)$ are constants which depend only on γ, δ. Using this estimate, we can prove the lemma (see [1, Lemma 5]).

Proof of the Theorem. Put $T=(A^2, B)$, $c_n=T^{-n/2}a_n$, $A_1=A/\sqrt{T}$, and $B_1=B/T$; then

$$c_n=A_1c_{n-1}+B_1c_{n-2}$$

holds with $c_0=0$ and $c_1=a_1/\sqrt{T}$. So c_n is written in the form

$$c_n=\frac{a_1}{\sqrt{T}}\frac{\alpha_1^n-\beta_1^n}{\alpha_1-\beta_1},$$

with $\alpha_1=\alpha/\sqrt{T}$ and $\beta_1=\beta/\sqrt{T}$. Noting $(A_1^2, B_1)=1$, we denote by L_n the Lehmer numbers associated with α_1, β_1. From the definition of L_n, we have

$$a_n=\begin{cases}a_1T^{(n-1)/2}L_n & \text{for } n \text{ odd},\\ Aa_1T^{n/2-1}L_n, & \text{for } n \text{ even}.\end{cases} \tag{2}$$

Now we prove

$$\log\,[a_1, a_2, ..., a_n]=\frac{3\log|\alpha_1|}{\pi^2}n^2+O\left(\frac{n^2}{\log n}\right), \tag{3}$$

and

$$\log|a_1a_2\cdots a_n| =\left(\frac{\log|\alpha_1|}{2}+\frac{\log T}{4}\right)n^2+O(n\log n). \tag{4}$$

It suffices to show when n is even.

From (2), we have

$$\begin{aligned}&\log\,[L_1, L_2, ..., L_{2m}]\\ &\quad\leqslant\log\,[a_1, a_2, ..., a_{2m}]\\ &\quad\leqslant\log|a_1AT^{m-1}|+\log\,[L_1, L_2, ..., L_{2m}].\end{aligned}$$

Using Lemma 1, we obtain (3). By Lemma 2

$$\begin{aligned}\log|a_1 a_2 \cdots a_{2m}| &= \log|a_1^{2m} A^m T^{m(m-1)}| + \log|L_1 L_2 \cdots L_{2m}| \\ &= \frac{\log|\alpha_1|}{2}(2m)^2 + m^2 \log T + O(2m \log 2m).\end{aligned}$$

Therefore we have (4).

Putting these estimates together, we get

$$\begin{aligned}\left(\frac{6 \log|a_1 a_2 \cdots a_n|}{\log[a_1, a_2, ..., a_n]}\right)^{1/2} &= \left[\frac{6(\log|\alpha_1|/2 + \log T/4)\, n^2 + O(n \log n)}{(3 \log|\alpha_1|/\pi^2)\, n^2 + O(n^2/\log n)}\right]^{1/2} \\ &= \pi\left(1 + \frac{\log T}{2 \log|\alpha_1|}\right)^{1/2} + O\left(\frac{1}{\log n}\right),\end{aligned}$$

with $\alpha_1 = \alpha/\sqrt{T}$. This completes the proof.

REFERENCES

1. P. KISS AND F. MÁTYÁS, An asymptotic formula for π, *J. Number Theory* **31** (1989), 255–259.
2. P. KISS, Primitive divisors of Lucas numbers, *in* "Application of Fibonacci Numbers" (A. N. Philipou *et al.*, Eds.), pp. 29–38, Kluwer, Hingham, MA, 1988.
3. D. H. LEHMER, An extended theory of Lucas functions, *Ann. of Math.* **31** (1930), 419–448.
4. C. L. STEWART, On divisors of Fermat, Fibonacci, Lucas and Lehmer numbers, *Proc. London Math. Soc.* **35** (1977), 425–447.

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