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# A New Type of Inclusion Exclusion Principle for Sequences and Asymptotic Formulas for $\zeta(k)$

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Let $\{a(n)\}_{n=0}^{\infty}$ be a sequence defined by the recurrence $a(n+2)=Aa(n+1)+Ba(n)$, where $a(0)=0$ and $a(1)$, $A$, $B$ are non-zero integers satisfying $A^2+4B\neq 0$. Assume that the ratio two roots of $x^2-Ax-B$ is not a root of unity. It is proved, for each positive integer $l$, that

$$\log\frac{|a(1^l)\,a(2^l)\cdots a(n^l)|}{[a(1^l), a(2^l), ..., a(n^l)]}=\frac{\zeta(l+1)}{1-\kappa}+O\left(\frac{\omega(n)}{n}\right),$$

where

$$\omega(n)=\begin{pmatrix}\log n & \text{for } l=1\\ 1 & \text{for } l\geqslant 2\end{pmatrix}$$

and $\kappa$ is a constant, depends only on $A$ and $B$. This formula is an improvement and a generalization of the results of S. Akiyama (1990, *J. Number Theory* **36**, 328–331). In the proof of this formula, we propose a new type of inclusion exclusion principle in a multiplicative sense which itself is of interest. 

## 1. INTRODUCTION

Let $\{a(n)\}_{n=0}^{\infty}$ be a sequence defined by the recurrence

$$a(n+2)=Aa(n+1)+Ba(n),$$

where $a(0)=0$ and $A$, $B$, $a(1)$ are non-zero integers satisfying $A^2+4B\neq 0$. Denote by $\alpha$, $\beta$ ($|\alpha|\geqslant|\beta|$) the roots of $x^2-Ax-B$. We also assume that $\alpha/\beta$ is not a root of unity, to assure $a(n)\neq 0$ for $n>0$. When $(A, B)=1$, then $\{a(n)\}_{n=0}^{\infty}$ is called a Lucas sequence. P. Kiss and F. Mátyás [4] showed that

$$\frac{\log|a(1)\,a(2)\cdots a(n)|}{\log[a(1), a(2), ..., a(n)]}=\zeta(2)+O\left(\frac{1}{\log n}\right), \tag{1}$$



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where $[u_1, u_2, ..., u_n]$ denotes the least common multiple of the terms $u_1, u_2, ..., u_n$ and $\zeta(s)$ the Riemann zeta function. It is well known that $\zeta(2)=\pi^2/6$. The condition $(A, B)=1$ was removed by the author [1], showing

$$\frac{\log |a(1)\, a(2) \cdots a(n)|}{\log[a(1), a(2), ..., a(n)]} = \frac{\zeta(2)}{1-\kappa} + O\left(\frac{1}{\log n}\right), \tag{2}$$

where $\kappa = \log((A^2, B))/(2 \log |\alpha|)$.

In the present paper, we improve the error term of (1) and (2) to $O((\log n)/n)$. Moreover we show, for each integer $l>1$, that

$$\frac{\log |a(1^l)\, a(2^l) \cdots a(n^l)|}{\log[a(1^l), a(2^l), ..., a(n^l)]} = \frac{\zeta(l+1)}{1-\kappa} + O\left(\frac{1}{n}\right). \tag{3}$$

Altough these results are concrete in nature, we employ an axiomatic description in Section 2. The reader will soon find that the basic idea lies in the purely elementary Theorem 1, which is what we call an "inclusion exclusion principle for sequences" in a multiplicative sense. Let us explain our idea roughly. Let $\{c(n)\}_{n=1}^{\infty}$ be the non-zero integer sequence with the property

$$m \mid n \rightarrow c(m) \mid c(n). \tag{P}$$

Then the division $c(n)/c(m)$ can be seen as the result of ruling out the older factor $c(m)$ from $c(n)$. If the sequence $\{c(n)\}_{n=1}^{\infty}$ admits the "inclusion exclusion principle," then the value

$$M(n) = \prod_{d \mid n} c(n/d)^{\mu(d)}$$

should be the "proper" new factor appearing in $c(n)$. Thus we have

$$[c(1), c(2), ..., c(n)] = \prod_{i \leqslant n} M(i). \tag{!}$$

This ideal situation is achieved by axioms (A1) and (A2) in Section 2. The Lucas sequence is a good example of a sequence which has these properties.

In Section 3, we evaluate $\log[c(1), c(2), ..., c(n)]$ from the estimation of $c(n)$, using Theorem 1. From this evaluation, we have no difficulty in obtaining a formula like (3).

In Section 4, we show another example of asymptotics formulas, which is not treated under the idea of Theorem 1.



*Notations*

| | |
|---|---|
| $\mathbb{N}$ | The set of positive integers. |
| $\mathrm{ord}_p(n)$ | The multiplicities of the prime $p$ when the integer $n$ is decomposed into prime factors. If $n=n_1/n_2$ is a rational number then define $\mathrm{ord}_p(n)$ by $\mathrm{ord}_p(n_1)-\mathrm{ord}_p(n_2)$. |
| $\varphi(\cdot)$ | Euler's totient function. |
| $\|x\|$ | The absolute value of $x$. |
| $[x]$ | The greatest integer not exceeding $x$. |

Let $f$ be a function from $\mathbb{N} \cup \{0\}$ to itself. We say $f$ is weakly increasing if $a \leqslant b$ implies $f(a) \leqslant f(b)$. We write $a \mid b$ if $b$ is divisible by $a$, and $a \nmid b$ if not.

## 2. The Key Theorem

Let $\{c(n)\}_{n=1}^{\infty}$ be a non-zero integer sequence. We consider two conditions:

(A1) For each prime $p$, we denote by $S_p$ the set of positive integer $n$'s so that $c(n)$ is divisible by $p$. If $S_p \neq \phi$, there exists an integer $r(p)$ and $S_p$ coincides with the set of positive $r(p)$ multiples.

(A2) There exists a weakly increasing function $f$ from $\mathbb{N} \cup \{0\}$ to itself with the property $\mathrm{ord}_p(c(n)) = f(\mathrm{ord}_p(n))$, for $n \in S_p$. (Especially, $\mathrm{ord}_p(c(n))$ is determined only by $\mathrm{ord}_p(n)$.)

Note that (A1) and (A2) imply property (P) introduced in Section 1. For a completely multiplicative function $m \rightarrow c(m)$, axioms (A1) and (A2) are not always satisfied, while (P) is trivial. We mention several examples with axioms (A1) and (A2).

Example 1. Let $l \in \mathbb{N} \cup \{0\}$ and $c(n)=n^l$. It is obvious that $\{n^l\}_{n=1}^{\infty}$ satisfies (A1) and (A2). It is easily seen that, up to a constant factor, these are the only polynomials with these properties.

Example 2. Let $\lambda$ be a weakly increasing function from $\mathbb{N} \cup \{0\}$ to itself. When $n \in \mathbb{N}$ is decomposed into prime factors as $\prod_i p_i^{e_i}$, we define $\rho(n) = \prod_i p_i^{\lambda(e_i)}$. Then we see that the sequence $\{\rho(n)\}_{n=1}^{\infty}$ satisfies (A1) and (A2).

Example 3. Let $c(n)=a(n)$, where $a(n)$ is a sequence defined in Section 1. If $(A, B)=1$ then $\{a(n)\}_{n=1}^{\infty}$ satifies (A1) and (A2). (This is the reason why we put $a(0)=0$.) Further, let $\gamma$ and $\delta$ be complex numbers satisfying

$$(\gamma+\delta)^2=C, \qquad \gamma\delta=D,$$

where $C, D$ are mutually coprime non-zero integers and $\gamma/\delta$ is not a root of unity. Then the Lehmer sequence $\{L(n)\}_{n=1}^{\infty}$ is defined by

$$L(n)=\begin{cases}(\gamma^n-\delta^n)/(\gamma^2-\delta^2) & \text{for } n \text{ even},\\ (\gamma^n-\delta^n)/(\gamma-\delta) & \text{for } n \text{ odd}.\end{cases}$$

It is proved in [6, Theorem 1.6] that $\{L(n)\}_{n=1}^{\infty}$ satisfies (A1) and (A2), which can easily be verified again by binomial expansion.

*Remark* 1. In [4, Lemma 1], it is asserted that the function $f$ corresponding to the axiom (A2) of the Lucas sequence $\{a(n)\}_{n=1}^{\infty}$ is

$$f(x)=x+\operatorname{ord}_p a(r(p)),$$

for any prime $p$. This is true when $p>2$. But when $p=2$, it is false. In this case, we have

$$f(x)=\begin{cases}\operatorname{ord}_2 a(r(2)) & \text{for } x=0,\\ \operatorname{ord}_2 a(2r(2))+x-1 & \text{for } x\geqslant 1.\end{cases}$$

For example, take $A=B=a(1)=1$ (the Fibonacci sequence). Then we have $a(6)=8$ while $a(3)=2$. But this minor fact does not cause any change of the results of [4].

PROPOSITION 1. *Let $\{c_1(n)\}_{n=1}^{\infty}$ and $\{c_2(n)\}_{n=1}^{\infty}$ be two sequences with properties* (A1) *and* (A2). *Then the composition sequence $\{c_1(c_2(n))\}_{n=1}^{\infty}$ also has properties* (A1) *and* (A2).

*Proof.* Let $p$ be a prime, and $c_1(c_2(n))$ be divisible by $p$ for some $n$. Let $r_i(p)$ $(i=1,2)$ be the positive integer determined by $\{c_i(n)\}_{n=1}^{\infty}$ $(i=1,2)$, respectively, by axiom (A1). Put

$$e_i(p)=\operatorname{ord}_p(r_i(p)) \qquad (i=1,2).$$

Then we have

$$p\mid c_1(c_2(n)) \leftrightarrow r_1(p)\mid c_2(n)$$
$$\leftrightarrow p_i^{e_i}\mid c_2(n) \qquad \text{for} \quad i=1,...,u,$$



where $r_1(p)=\prod_{i=1}^{u} p_i^{e_i}$. Using (A1) and (A2), the last statement is equivalent to

$$r_2(p_i)\,p_i^{\max\{e_i-e_2(p_i),0\}}\mid n \qquad \text{for} \quad i=1,...,u$$
$$\leftrightarrow \text{LCM of terms } r_2(p_i)\,p_i^{\max\{e_i-e_2(p_i),0\}}\mid n.$$

Thus we have proved axiom (A1) for $\{c_1(c_2(n))\}_{n=1}^{\infty}$. Let $f_i$ be the function determined by $\{c_i(n)\}_{n=1}^{\infty}$ $(i=1,2)$, respectively, by axiom (A2). Considering the composition $f_1\circ f_2$, we easily see that (A2) is satisfied by $\{c_1(c_2(n))\}_{n=1}^{\infty}$.

The next proposition, which we do not use in the next section, provides interesting sequences with properties (A1) and (A2) as well as Proposition 1.

PROPOSITION 2. *Let $\{c_1(n)\}_{n=1}^{\infty}$ and $\{c_2(n)\}_{n=1}^{\infty}$ be two sequences with properties* (A1) *and* (A2). *Then the sequence of greatest common divisors $\{(c_1(n), c_2(n))\}_{n=1}^{\infty}$ also has the same properties.*

*Proof.* We use the notation of the proof of Lemma 1. Then we have

$$p\mid(c_1(n),c_2(n)) \leftrightarrow [r_1(p),r_2(p)]\mid n,$$
$$\operatorname{ord}_p((c_1(n),c_2(n)))=\min\{\operatorname{ord}_p c_1(n), \operatorname{ord}_p c_2(n)\}.$$

Hence axioms (A1) and (A2) hold.

Now we prove our key theorem.

THEOREM 1. *Let $\{c(n)\}_{n=1}^{\infty}$ be a non-zero integer sequence with properties* (A1) *and* (A2). *Define $M(i)$ by*

$$M(i)=\prod_{d\mid i} c(i/d)^{\mu(d)},$$

*where $\mu(\cdot)$ is the Möbius function. Then $\{M(i)\}_{i=1}^{\infty}$ is an integer sequence and we have*

$$[c(1),c(2),...,c(n)]=\prod_{i=1}^{n}|M(i)|.$$

*Proof.* First, let us calculate $\operatorname{ord}_p M(i)$. Using (A1), we have

$$\operatorname{ord}_p M(i)=\sum_{d\mid i,\ r(p)\mid i/d}\mu(d)\operatorname{ord}_p c(i/d)$$
$$=\sum_{d\mid i/r(p)}\mu(d)\operatorname{ord}_p c(i/d).$$

Regard the last sum as 0 when $i/r(p)$ is not an integer. With division into parts by $p$ divisibility, the sum becomes

$$\sum_{d \mid i/r(p)\, p} \mu(pd) \operatorname{ord}_p c(i/pd) + \sum_{d \mid i/r(p),\, (d,\, p)=1} \mu(d) \operatorname{ord}_p c(i/d).$$

When $i$ is not a multiple of $r(p)\, p$, then the first sum vanishes. As the value of the Möbius function $\mu(n)$ is zero when $n$ is not square free, the sum becomes

$$\sum_{\substack{d \mid i/r(p) \\ (d,\, p)=1}} \mu(d) \operatorname{ord}_p c(i/d) - \sum_{\substack{d \mid i/r(p)\, p \\ (d,\, p)=1}} \mu(d) \operatorname{ord}_p c(i/pd).$$

Let

$$i/r(p) = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r} \qquad \text{and} \qquad p_1 = p,$$

where $p_i$ $(i = 1, 2, ..., r)$ is a prime and $e_i \in \mathbb{N} \cup \{0\}$, be the prime decomposition. For the simplicity, put $e = e_1$. (To express the case $i = r(p)$ or the case $i = r(p)\, p^e$, we use $\mathbb{N} \cup \{0\}$ instead of $\mathbb{N}$.) Then we have

$$\operatorname{ord}_p M(i) = \sum_{d \mid p_2 \cdots p_r} \mu(d) \operatorname{ord}_p c(i/d) - \delta \sum_{d \mid p_2 \cdots p_r} \mu(d) \operatorname{ord}_p c(i/pd),$$

where

$$\delta = \begin{cases} 1 & \text{if } \ r(p)\, p \mid i \\ 0 & \text{if } \ r(p)\, p \nmid i. \end{cases}$$

By axiom (A2), we see that $\operatorname{ord}_p c(i/d)$ and $\operatorname{ord}_p(i/pd)$ are invariant with respect to $d$. Thus we have

$$\operatorname{ord}_p M(i) = \delta_1 \operatorname{ord}_p c(i) - \delta_1 \operatorname{ord}_p c(i/p),$$

where

$$\delta_1 = \begin{cases} 1 & \text{if } \ i = r(p)\, p^e, \\ 0 & \text{otherwise.} \end{cases}$$

We regard $\operatorname{ord}_p c(i/p)$ as 0 when $i/p$ is not an integer. From this formula and axiom (A2), we see that $\operatorname{ord}_p M(i) \geqslant 0$ for any prime $p$. This shows that $M(i)$ is an integer.

Next we consider $\operatorname{ord}_p([c(1), ..., c(i)]/[c(1), ..., c(i-1)])$, which we denote by $w(i)$ $(i \geqslant 2)$.



*Case* 1 (when $r(p)\, p^e \mid i$ and $e > 0$). If $r(p)\, p^e \mid i$ $(e > 0)$ and $i \neq r(p)\, p^e$ then $w(i) = 0$, because if $i = r(p)\, kp^e$, $k \geqslant 2$, and $(k, p) = 1$ then

$$c(r(p)\, p^e) \mid [c(1), ..., c(i-1)]$$

and $\operatorname{ord}_p c(r(p)\, kp^e) = \operatorname{ord}_p c(r(p)\, p^e)$ by (A2). So we must consider the case $i = r(p)\, p^e$. Using (A1) and (A2), we see that if $\operatorname{ord}_p c(r(p)\, p^e) > \operatorname{ord}_p c(r(p)\, p^{e-1})$ then $i = r(p)\, p^e$ is the first index for which $c(i)$ is divisible by $p^{\operatorname{ord}_p c(r(p)\, p^e)}$. Thus we have $w(r(p)\, p^e) = \operatorname{ord}_p c(r(p)\, p^e) - \operatorname{ord}_p c(r(p)\, p^{e-1})$. If $\operatorname{ord}_p c(r(p)\, p^e)$ is equal to $\operatorname{ord}_p c(r(p)\, p^{e-1})$, then

$$\operatorname{ord}_p[c(1), ..., c(i-1)] = \operatorname{ord}_p[c(1), ..., c(i)],$$

and we have $w(i) = 0$.

*Case* 2 (otherwise). In this case, if $i \neq r(p)$ we see that $w(i) = 0$ by a reason similar to that of Case 1. And we have $w(r(p)) = \operatorname{ord}_p(c(r(p)))$, because $i = r(p)$ is the first index for which $c(i)$ is divisible by $p$.

Summing up, we have shown, for $i \geqslant 2$,

$$\operatorname{ord}_p[c(1), ..., c(i)]/[c(1), ..., c(i-1)] = \operatorname{ord}_p M(i),$$

for any prime $p$. By definition, we have $M(1) = c(1)$. This completes the proof by induction.

## 3. The Applications of Theorem 1

Suppose we have enough precise information about the asymptotic behavior of $c(n)$ or $M(n)$. Then we can evaluate the least common multiple $[c(1), ..., c(n)]$, using Theorem 1.

Proposition 3. *Let $l$ be a positive integer and $\{c(n)\}_{n=1}^{\infty}$ be a non-zero integer sequence with properties* (A1) *and* (A2) *in Theorem* 1. *Assume that there exist constant $C_i$ $(i = 1, ..., l)$ such that*

$$\log |c(n)| = C_l n^l + C_{l-1} n^{l-1} + \cdots + C_1 n + O(\log n),$$

*where $C_l \neq 0$. Then we have*

$$\log[c(1), ..., c(n)] = C_l \frac{n^{l+1}}{(l+1)\, \zeta(l+1)} + O(n^l \omega(n)),$$

*where*

$$\omega(n)=\begin{cases}\log n & \text{for } l=1\\ 1 & \text{for } l\geqslant 2.\end{cases}$$

*Proof.* Using Theorem 1, we must evaluate

$$\begin{aligned}\log[c(1), ..., c(n)] &= \sum_{i=1}^{n} \log|M(i)|\\ &= \sum_{i=1}^{n}\sum_{d\mid i}\mu(d)\log|c(i/d)|\\ &= \sum_{d=1}^{n}\mu(d)\sum_{k\leqslant n/d}\log|c(k)|.\end{aligned}$$

Put $P(n)=C_l n^l + C_{l-1}n^{l-1}+\cdots+C_1 n$ and $Q(n)=\log|c(n)|-P(n)$. Separate the last sum into two parts:

$$\sum_{d=1}^{n}\mu(d)\sum_{k\leqslant n/d}P(k)+\sum_{d=1}^{n}\mu(d)\sum_{k\leqslant n/d}Q(k).$$

We see easily see that

$$\begin{aligned}\sum_{k\leqslant n/d}P(k) &= C_l\sum_{k=1}^{[n/d]}k^l+O((n/d)^l)\\ &= C_l\frac{(n/d)^{l+1}}{l+1}+O((n/d)^l).\end{aligned}$$

So the first sum becomes

$$\begin{aligned}& C_l\frac{n^{l+1}}{l+1}\sum_{d=1}^{n}\frac{\mu(d)}{d^{l+1}}+O\left(n^l\sum_{d=1}^{n}\frac{1}{d^l}\right)\\ &= C_l\frac{n^{l+1}}{(l+1)\,\zeta(l+1)}+O(n^l\omega(n)).\end{aligned}$$

Here we used the estimation

$$\frac{1}{\zeta(l+1)}=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^{l+1}}=\sum_{d=1}^{n}\frac{\mu(d)}{d^{l+1}}+O(n^{-l}).$$

Thus to prove Proposition 3, it suffices to show that

$$\sum_{d=1}^{n}\mu(d)\sum_{k\leqslant n/d}Q(k)=O(n\log n). \tag{4}$$



If we change the order of the summation, the left hand side becomes

$$\sum_{k=1}^{n}Q(k)\sum_{d\leqslant n/k}\mu(d).$$

It is known that there exists a positive constant $v$ such that

$$\sum_{d\leqslant x}\mu(d)=O(x\exp(-v\sqrt{\log x})).$$

By the assumption, we have

$$\begin{aligned}\sum_{k=1}^{n}Q(k)\sum_{d\leqslant n/k}\mu(d) &= \sum_{k>n/2}^{n}Q(k)+\sum_{k=1}^{n/2}Q(k)\sum_{d\leqslant n/k}\mu(d)\\ &= \sum_{k>n/2}^{n}O(\log k)\\ &\quad+O\left(\sum_{k=1}^{n/2}\log k\frac{n/k}{\exp(v\sqrt{\log(n/k)})}\right).\\ &= O(n\log n)\\ &\quad+O\left(n\int_1^{n/2}\frac{\log u}{u}\frac{du}{\exp(v\sqrt{\log(n/u)})}\right).\end{aligned}$$

So it remains to prove

$$\int_1^{n/2}\frac{\log u}{u}\frac{du}{\exp(v\sqrt{\log(n/u)})}=O(\log n). \tag{5}$$

If we change the variable, the left hand side is equal to

$$\begin{aligned}&\int_n^2\frac{\log(n/v)}{n/v}\frac{n(-v^{-2})}{\exp(v\sqrt{\log v})}dv\\ &=\log n\int_2^n\frac{dv}{v\exp(v\sqrt{\log v})}-\int_2^n\frac{\log v}{v\exp(v\sqrt{\log v})}dv.\end{aligned}$$

The last two integrals are bounded when $n$ tends to infinity, because

$$\exp(-v\sqrt{\log v})=O(1/(\log v)^3).$$

This completes the proof.

*Remark* 2. To prove (4), we followed the argument of P. Kiss [5]. It seems that there is a minor error evaluating the integral (5) in [5], which is corrected in the above way.

Now, let us state the result, which implies the improvement of the assertions (1) and (2) of the Introduction.

THEOREM 2. *Let* $\{c(n)\}_{n=1}^{\infty}$ *be a non-zero integer sequence with axioms* (A1) *and* (A2), *which has the asymptotic behavior*

$$\log|c(n)| = C_l n^l + C_{l-1} n^{l-1} + \cdots + C_1 n + O(\log n),$$

*where* $C_i$ $(i=1, ..., l)$ *are constants and* $C_l \neq 0$. *Then we have*

$$\frac{\log|c(1)\,c(2)\cdots c(n)|}{\log[c(1), c(2), ..., c(n)]} = \zeta(l+1) + O\left(\frac{\omega(n)}{n}\right),$$

*where* $\omega(n)$ *is defined in Proposition* 3.

*Proof.* We easily see that

$$\log|c(1)\,c(2)\cdots c(n)| = C_l \frac{n^{l+1}}{l+1} + O(n^l\omega(n)).$$

Using the estimation of $\log[c(1), c(2), ..., c(n)]$ in Proposition 3, we have the assertion.

COROLLARY 1. *Let* $\{c(n)\}_{n=1}^{\infty}$ *be the Lucas sequence or the Lehmer sequence defined in Section* 2, *Example* 3. *Then we have*

$$\frac{\log|c(1)\,c(2)\cdots c(n)|}{\log[c(1), c(2), ..., c(n)]} = \zeta(2) + O\left(\frac{\log n}{n}\right).$$

*Proof.* We prove the case of the Lucas sequence. By definition, we have

$$c(n) = c(1)\frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha$, $\beta$ $(|\alpha| \geqslant |\beta|)$ are complex numbers and $\alpha/\beta$ is not a root of unity. So

$$\log|c(n)| = n\log|\alpha| + \log|1-(\beta/\alpha)^n| + \log|c(1)/(\alpha-\beta)|.$$

If $|\alpha|$ is strictly bigger than $|\beta|$, then it is seen that

$$\log|1-(\beta/\alpha)^n| = O((\beta/\alpha)^n).$$

In the case $|\alpha| = |\beta|$, using the result of Baker on the summation of the logarithm of algebraic numbers, we see that

$$|1-(\alpha/\beta)^n| > n^{-c}$$



for some positive constant $c$ because $\alpha/\beta$ is not a root of unity (see [5]). At any rate, we have

$$\log|c(n)| = n\log|\alpha| + O(\log n).$$

Thus we can apply Theorem 2. For the Lehmer sequence, the proof is almost the same, so we omit it.

In Corollary 1, above in the case $|\alpha| > |\beta|$, we have

$$\log c(n) = n\log|\alpha| + \log|c(1)/(\alpha-\beta)| + O(|\beta/\alpha|^n).$$

Using the notation of Proposition 3, we have

$$\log[c(1), ..., c(n)] = \sum_{d=1}^{n} \mu(d) \sum_{k \leqslant n/d} P(k) + \sum_{d=1}^{n} \mu(d) \sum_{k \leqslant n/d} Q(k),$$

where $P(k) = k\log|\alpha|$ and $Q(k) = \log|c(1)/(\alpha-\beta)| + O(|\beta/\alpha|^k)$. From this, we easily have

$$\log[c(1), ..., c(n)] = \log|\alpha| \sum_{m=1}^{n} \sum_{dk=m} \mu(d)\,k + O(n)$$
$$= \log|\alpha| \sum_{m=1}^{n} \varphi(m) + O(n).$$

And we also have

$$\log|c(1)\cdots c(n)| = \log|\alpha|\frac{n^2}{2} + O(n).$$

Put

$$E(n) = \sum_{m=1}^{n} \varphi(m) - \frac{3}{\pi^2}n^2. \qquad (6)$$

The estimation of $E(n)$ is a classical problem. S. S. Pallai and S. Chowla [7] showed that

$$E(n) = \Omega(n\log\log\log n).$$

Thus we have

COROLLARY 2. *Let* $\{c(n)\}_{n=1}^{\infty}$ *be the Lucas sequence or the Lehmer* *sequence defined in Section 2, Example 3. If the absolute values of the characteristic roots which define the sequence $\{c(n)\}_{n=1}^{\infty}$ are distinct, we have*

$$\frac{\log |c(1)\, c(2) \cdots c(n)|}{\log [c(1), c(2), ..., c(n)]} = \zeta(2) + O\left(\frac{E(n)}{n^2}\right),$$

*where $E(n)$ is defined by (6).*

It seems that the best estimation of $E(n)$ up to now is

$$E(n) = O(n \log^{2/3} n (\log \log n)^{4/3})$$

due to A. Walfisz [8]. Applying this estimate to Corollary 2 gives a conditional improvement of Corollary 1. It seems interesting that the problem of the estimation of the least common multiple of Lucas and Lehmer numbers also needs deep results of analytic number theory.

Finally, in this section, we show the asmptotic formula of $\zeta(k)$ cited in the Introduction.

THEOREM 3. *Let $\{a(n)\}_{n=0}^{\infty}$ be the sequence defined by the recurrence*

$$a(n+2) = Aa(n+1) + Ba(n),$$

*where $a(0)=0$ and $A$, $B$, $a(1)$ are non-zero integers satisfying $A^2+4B \neq 0$. Denote by $\alpha$, $\beta$ ($|\alpha| \geqslant |\beta|$) the roots of $x^2 - Ax - B$. Assume that $\alpha/\beta$ is not a root of unity. Then we have, for an integer $l$,*

$$\frac{\log |a(1^l)\, a(2^l) \cdots a(n^l)|}{\log[a(1^l), a(2^l), ..., a(n^l)]} = \frac{\zeta(l+1)}{1-\kappa} + O\left(\frac{\omega(n)}{n}\right),$$

*where $\omega(n)$ is defined in Proposition 3 and $\kappa = \log((A^2, B))/\log |\alpha|^2$.*

*Remark* 3. We easily see that the condition of $\alpha/\beta$ implies $0 \leqslant \kappa < 1$.

*Proof.* We follow the argument of the author [1] to deduce the problem of Proposition 3. Let $T=(A^2, B)$, $\alpha_1 = \alpha/\sqrt{T}$ and $\beta_1 = \beta/\sqrt{T}$. Then we have

$$a(n) = \begin{cases} a(1)\, T^{(n-1)/2} L(n) & \text{for } n \text{ odd}, \\ Aa(1) T^{n/2-1} L(n) & \text{for } n \text{ even}, \end{cases}$$

where $L(n)$ is the Lehmer sequence associated to $\alpha_1$, $\beta_1$ (see the formula (2) of [1]). By Proposition 1, the sequence $\{L(n^l)\}_{n=1}^{\infty}$ satisfies axioms (A1) and (A2) and has the asymptotic behavior

$$\log |L(n^l)| = n^l \log |\alpha_1| + O(\log n).$$



Here we used the argument of Corollary 1. From Proposition 3,

$$\log [L(1^l), ..., L(n^l)] = \frac{\log |\alpha_1|}{(l+1)\ \zeta(l+1)} n^{l+1} + O(n^l \omega(n)).$$

Using arguments similar to those of [1], we have

$$\log[a(1^l), ..., a(n^l)] = \frac{\log |\alpha_1|}{(l+1)\, \zeta(l+1)} n^{l+1} + O(n^l \omega(n))$$

and

$$\log |a(1^l) \cdots a(n^l)| = \frac{\log |\alpha_1|}{l+1} n^{l+1} + \frac{\log T}{2(l+1)} n^{l+1} + O(n^l \omega(n)).$$

Putting these estimates together, we obtain the result.

*Remark* 4. We see that Theorem 3 implies Corollary 1. And if $(A, B)=1$ then $\kappa=0$ and $\{a(n)\}_{n=1}^{\infty}$ is a Lucas sequence, and we also see that Theorem 3 is an immediate consequence of Theorem 2.

*Remark* 5. We can also treat the sequence $\{a(sn^l)\}_{n=1}^{\infty}$, where $s$ is a non-zero integer. But this is not of interest because we have

$$a(sn^l) = b(n^l)\, a(s),$$

where $b(n)$ is a sequence of the type described in Theorem 3 correponding to the characteristic roots $\alpha^s$ and $\beta^s$.

*Remark* 6. In the case $l=1$ and $|\alpha| > |\beta|$, the error term of Theorem 3 can be replaced by $O(E(n)/n^2)$, using arguments similar to those of Corollary 2.

## 4. EXAMPLES AND CONCLUDING REMARKS

We have treated the sequences with axioms (A1) and (A2) so far. To generalize these arguments, the following problems should be answered.

(I) Is there any other sequence with properties (A1) and (A2) and with a good asymptotic behavior?

(II) How can we characterize the sequence $\{c(n)\}_{n=1}^{\infty}$, which has a good asymptotic behavior and for which

$$[c(1), ..., c(n)] = \prod_{i=1}^{n} |M(i)|$$

holds? Similarly, what about the case

$$[c(1), ..., c(n)] \doteqdot \prod_{i=1}^{n} |M(i)| ? \tag{7}$$

In this section, we demonstrate that there are some sequences which have property (7). We do not have satisfactory answers to problems (I) and (II).

Let $\{a(n)\}_{n=1}^{\infty}$ be the Lucas sequences defined in Section 1. Put $b(n) = a(tn)$, where $t$ is a positive integer greater than 1. By Proposition 1, the sequence $\{b(n)\}_{n=1}^{\infty}$ also has properties (A1) and (A2). Define $c(n) = a(n)\, b(n)$. The sequence $\{c(n)\}_{n=1}^{\infty}$ does not satisfy axioms (A1) and (A2).

THEOREM 4. *With notation as above, we have*

$$\log[c(1), ..., c(n)] = \log[a(1), ..., a(n)][b(1), ..., b(n)] + O(n).$$

From this theorem, we have

$$\frac{\log |c(1) \cdots c(n)|}{\log[c(1), ..., c(n)]} = \zeta(2) + O\left(\frac{\log n}{n}\right),$$

for example. The proof of Theorem 4 is elementary but somewhat complicated, especially for two power divisibility. We omit the proof. We only mention that there is an explicit formula for

$$[a(1), ..., a(n)][b(1), ..., b(n)]/[c(1), ..., c(n)]. \tag{8}$$

In another direction, J. P. Jones and P. Kiss [3] treated the sequence of type $\{b(n)/a(n)\}_{n=1}^{\infty}$ with notation as above, which is a generalization of the results of J. P. Bézivin [2]. They derived an asymptotic fomula of similar type.

*Note.* The idea of Theorem 4 came from the following example, which I was informed of by Y. Tanigawa. Define $\{a(n)\}_{n=1}^{\infty}$ and $\{b(n)\}_{n=1}^{\infty}$ by the recurrence

$$a(n+2) = a(n+1) + a(n), \qquad b(n+2) = 11b(n+1) + b(n),$$

where $a(0) = b(0) = 0$ and $a(1) = b(1) = 1$. Let

$$g(n) = [a(1), ..., a(n)][b(1), ..., b(n)]/[a(1)\, b(1), ..., a(n)\, b(n)].$$

Tanigawa then found that

$$g(n) = \begin{cases} 1 & \text{for } \; n \leqslant 21 \text{ or } 110 \leqslant n < 150, \\ 11 & \text{for } \; 22 \leqslant n < 110. \end{cases} \tag{!}$$



This example is the case $t = 5$ as stated above, up to a constant factor. Using the explicit formula mentioned above, we know that $g(n)$ is a square free integer and each prime factor is congruent to 1 modulo 10. And we have, for example,

$$\operatorname{ord}_{11} g(n) = \begin{cases} 1 & \text{for } \; 2 \cdot 11^e \leqslant n < 10 \cdot 11^e, \\ 0 & \text{otherwise}, \end{cases}$$

$$\operatorname{ord}_{31} g(n) = \begin{cases} 1 & \text{for } \; 6 \cdot 31^e \leqslant n < 30 \cdot 31^e, \\ 0 & \text{otherwise}, \end{cases}$$

$$\operatorname{ord}_{41} g(n) = \begin{cases} 1 & \text{for } \; 4 \cdot 41^e \leqslant n < 20 \cdot 41^e, \\ 0 & \text{otherwise}, \end{cases}$$

$$\operatorname{ord}_{61} g(n) = \begin{cases} 1 & \text{for } \; 3 \cdot 61^e \leqslant n < 15 \cdot 61^e, \\ 0 & \text{otherwise}, \end{cases}$$

where $e = 1, 2, ...$. Not every prime which is congruent to 1 modulo 10 appears as a factor of $g(n)$. The first two counter-examples are 211 and 281. The general explicit formula for (8) will be publised elsewhere.

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