Cubic Pisot units with finite beta expansions

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Abstract. Cubic Pisot units with finite beta expansion property are classified (Theorem 3). The results of [6] and [3] are well combined to complete its proof. Further, it is noted that the above finiteness property is equivalent to an important problem of fractal tiling generated by Pisot numbers.

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1. Introduction and the results

Let $\beta > 1$ be a fixed real number. Any positive real x is expanded as:

$$x = \sum_{i=N_0}^{\infty} a_{-i}\beta^{-i} = a_{-N_0}\beta^{-N_0} + a_{-N_0-1}\beta^{-N_0-1} + \cdots$$

with $a_i \in \mathbb{Z} \cap [0, \beta)$. Here we assume the 'greedy condition':

$$\left| x - \sum_{i=N_0}^{N} a_{-i}\beta^{-i} \right| < \beta^{-N} \tag{1}$$

holds for all $N \geq N_0$. Hereafter we call this expansion a *beta expansion* of x in base β. This is a natural extension of decimal or binary expansion to a real base β. Fundamental ergodic properties of this expansion, as a dynamical system on the real torus $\mathbb{R}/\mathbb{Z}$, can be found in Rényi [14] and Parry [11]. A *Pisot number* is an algebraic integer whose conjugates other than itself have modulus less than one. A *Salem number* is an algebraic integer whose conjugates other than itself have modulus less than or equal to one, and at least one conjugate has modulus one. Let $\mathrm{Fin}(\beta)$ be a set consisting of all finite beta expansions. Consider the condition

$$\text{(F)} \qquad \mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}.$$

It is easily seen that if $\beta > 1$ is an integer, then (F) holds. Conversely it is proved in Lemma 1 of [6], the condition (F) implies that β is a Pisot number. In the same paper, they showed that if $\mathbb{Z}_{\geq 0} \subset \mathrm{Fin}(\beta)$ then β must be a Pisot number or a Salem number. We can show a slight improvement of this.

Proposition 1. *Let $\beta > 1$ be a real number. Then $\mathbb{Z}_{\geq 0} \subset \mathrm{Fin}(\beta)$ implies that β is a Pisot number.*

From now on we assume that β is a Pisot number. There exists a Pisot number which does not satisfy (F). To find a simple algebraic characterization of Pisot numbers with (F) is an open question, to this date. Let

$$\mathrm{Irr}(\beta) = x^m - a_{m-1}x^{m-1} - a_{m-2}x^{m-2} - \ldots - a_0$$

be a minimal polynomial of β with $a_i \in \mathbb{Z}$. In [6], a beautiful sufficient condition is shown:

Theorem 1 (Frougny and Solomyak). *If*

$$a_{m-1} \geq a_{m-2} \geq \ldots \geq a_0 > 0,$$

then β has property (F).

Their proof is an algorithmic one, by showing the existence of finite algorithm to rewrite each element of $\mathbb{Z}[\beta^{-1}]_{\geq 0}$ into a form satisfying (1). M.Hollander, in his thesis [7], showed another sufficient condition:

$$a_{m-1} > a_{m-2} + a_{m-3} + \ldots + a_0 \qquad \text{with } a_i \geq 0.$$

The author proved a necessary and sufficient condition for a *fixed* β in [3] and [1]. We quote it, in a weak form:

Theorem 2 (Akiyama). *Let β be a Pisot number. Then β has the property* (F) *if and only if every element of*

$$\left\{ x \in \mathbb{Z}[\beta] \;\middle|\; 0 < x = x^{(1)} < 1, |x^{(j)}| \leq \frac{[\beta]}{1 - |\beta^{(j)}|} \quad j = 2, 3, \ldots, m \right\}$$

has finite beta expansion in base β. Here $x^{(i)}$ $(i = 1, 2, \ldots, m)$ are the conjugates of $x \in \mathbb{Q}(\beta)$.

Since the above set is finite, we can actually determine whether β has property (F) or not. Obvious defect of this result is the vagueness of the condition.

A Pisot number β is called a *Pisot unit* if it is also a unit of the integer ring of $\mathbb{Q}[\beta]$. The main purpose of this note is to compare above two theorems. As a result, combining these we can show,

Theorem 3. *Let β be a cubic Pisot number. Then $\beta > 1$ has property* (F) *if and only if β is a root of the following polynomial with integer coefficients:*

$$x^3 - ax^2 - bx - 1, \qquad a \geq 0 \text{ and } -1 \leq b \leq a + 1.$$

This theorem asserts that the Pisot number defined by

$$x^3 - ax^2 + x - 1, \quad a \geq 2,$$

and

$$x^3 - ax^2 - (a+1)x - 1, \quad a \geq 0,$$

has property (F). (Note that the first type of polynomials are not irreducible when $a = 0, 1$.) These are not included in the former results of [6] and [7], but the proof is rather lengthy and established by brute force. See section 3 and 4.

The beta expansion by a Pisot number has a close connection with tiling of the Euclidean space. In [13], G.Rauzy constructed a domain with a fractal boundary by the Pisot number β with $\mathrm{Irr}(\beta) = x^3 - x^2 - x - 1$. In [18], you can find a formulation of the tiling by such domains for general Pisot numbers. These results are extended by many authors. See e.g., [8] and [9]. The condition (F) is used to construct Markov partition for a certain toral automorphism in [12]. On the other hand, the author studied these tiling from a different point of view in [2] and [3]. Now we review this result. Let Φ be the map from $\mathbb{Q}(\beta)$ to $\mathbb{R}^{m-1}$ defined by:

$$\Phi(x) = (x^{(2)}, x^{(3)} \dots, x^{(r_1)}, \mathrm{Re}\, x^{(r_1+1)}, \mathrm{Im}\, x^{(r_1+1)}, \dots, \mathrm{Re}\, x^{(r_1+r_2)}, \mathrm{Im}\, x^{(r_1+r_2)})$$

where $\beta^{(i)} \quad (i = 1, 2, \dots, r_1)$ are real conjugates and $\beta^{(r_1+i)} \quad (i = 1, 2, \dots, r_2)$ and $\beta^{(r_1+r_2+i)} (= \overline{\beta^{(r_1+i)}}) \quad (i = 1, 2, \dots, r_2)$ are complex conjugates of β. Let us classify elements of $\mathrm{Fin}(\beta)$ by its fractional part:

$$\mathrm{Fin}(\beta) = \bigsqcup_{\omega} S_{\omega}.$$

Here ω runs through all possible fractional parts with (1) and S_{ω} is the subset of $\mathrm{Fin}(\beta)$ consisting of elements whose fractional part coincides with ω. Each element ω is expressed as a greedy word with the leading point character, for example .1 or .001. Especially $S_{.}$ is the set of all finite beta expansions in base β with no fractional part. Applying Φ and taking the closure by the topology of $\mathbb{R}^{m-1}$, we see

$$\mathbb{R}^{m-1} = \bigsqcup_{\omega} \overline{\Phi(S_{\omega})},$$

when β is a Pisot number with property (F), by Theorem 2 of [3]. Denote $T_{\omega} = \overline{\Phi(S_{\omega})}$. We quote Theorem 3 of [3] (the essential idea can be found in [1]).

Theorem (Akiyama). *Suppose β is a Pisot unit with property* (F). *Then the origin is an inner point of $T_{.} = \overline{\Phi(S_{.})}$.*

Concerning this theorem, we have to say a few words on the works of B.Praggastis. The author got to know by the paper of [17], she already showed this at least in a special case in her thesis [12]. It seems the strategy of two proofs are quite different. Her method is to construct a Markov partition in a general situation. Her construction seems a little involved. Our method is rather restricted but readers are required a quite simple geometry of numbers. The author hopes that this way is promising in studying the more precise information on this tiling.

In any case, this theorem is very much fundamental. In fact, we can show, the set of inner points of T_ω is dense in T_ω. (In the standard terminology, by this property, we can call T_ω to be a *tile*.) Moreover we see the boundary is nowhere dense in $\mathbb{R}^{m-1}$ (see [3]).

In this paper, we show

Proposition 2. *Let β be a Pisot unit. Then β has property* (F) *if and only if the origin in an inner point of $T_.$.*

This proposition is a certain geometric characterization of the property (F). It also describes the significance of our Theorem 3. The proofs of Proposition 1 and 2 are established in the last section.

2. Cubic Pisot units and their expansion of 1

For the moment let $\beta > 1$ be an arbitrary real number. Consider the beta expansion of the positive number:

$$0 < 1 - [\beta]\beta^{-1} = c_{-2}\beta^{-2} + c_{-3}\beta^{-3} + \ldots = .0c_{-2}c_{-3}\ldots .$$

Putting $c_{-1} = [\beta]$, we can formally write

$$1 = .c_{-1}c_{-2}c_{-3}\ldots .$$

This expansion $.c_{-1}c_{-2}c_{-3}\ldots$ is called the *expansion of* 1, which we denote by $d(1,\beta)$. We will identify this expression with the word $c_{-1}c_{-2}c_{-3}\ldots$ generated by $A = \mathbb{Z} \cap [0,\beta)$. Every finite word generated by A represents a beta expansion in base β if and only if the word is lexicographically less than $d(1,\beta)$ at any starting point. This fact can be generalized to infinite words apart from certain exceptions. See [11] for the details.

Now let $\beta > 1$ be a cubic number with

$$\mathrm{Irr}(\beta) = x^3 - ax^2 - bx - c,$$

and $c \neq 0$. Then we have

Lemma 1. *The number β is a Pisot number if and only if both*

$$|b-1| < a + c \text{ and } (c^2 - b) < \mathrm{sgn}(c)(1 + ac)$$

holds.

Proof. Let $f(x) = x^3 - ax^2 - bx - c$ and $f(\beta) = 0$ with $\beta > 1$. Then above conditions are equivalent to $f(\pm 1) < 0$ and $f(|c|) < 0$. First we show the necessity of these conditions. If $f(1) \geq 0$ or $f(-1) \geq 0$ then f has another root whose modulus is not less than 1. Thus β can not be a Pisot number. Since $|c|$ is the absolute norm of β, we see $\beta > |c|$ when β is a Pisot number. This implies $f(|c|) < 0$, by the similar consideration.

Now we show the sufficiency. Assume that $f(\pm 1) < 0$ and $f(|c|) < 0$. Firstly, assume that β is not totally real and β' is a non real conjugate of β. Then we see $|c| = |\beta\beta'\overline{\beta'}| = \beta|\beta'|^2$. Since β is the only real root of f, we see $f(|c|) < 0$ implies $\beta > |c|$. Thus β must be a Pisot number. Secondly, let β be totally real and β' and β'' be the real conjugates. Then $f(\pm 1) < 0$ implies that $|\beta'| - 1$ and $|\beta''| - 1$ are both positive, or both negative. If $|\beta'| > 1$ and $|\beta''| > 1$, then $|c| = \beta|\beta'\beta''|$ is greater than all roots of f. Thus $f(|c|) > 0$, which contradict with our assumption. We have have shown $|\beta'| < 1$ and $|\beta''| < 1$, which shows that β is a Pisot number. $\square$

By Proposition 1 of [1], if a Pisot number has property (F) then β does not have any positive conjugate other than β. This implies $c > 0$. Thus if we consider a Pisot unit with (F), we may assume $c = 1$. By Lemma 1, we see $|b - 1| \le a$ in this case, which implies $a \ge 0$.

Lemma 2. *Let $\beta > 1$ be a cubic Pisot number with $c = 1$. Then the expansion of 1 in base β is given by the following table.*

b	$d(1,\beta)$
$-a+1 \le b \le -2$	$a-1, a+b-1, \widetilde{a+b}$
$b = -1$	$a-1, a-1, 0, 1$
$0 \le b \le a$	$a, b, 1$
$b = a+1$	$a+1, 0, 0, a, 1$

Here $\widetilde{w}$ is the periodic expansion $w, w, w, \ldots$.

Proof. One can easily see the right hand side is formally equal to 1, by the minimal polynomial. According to the result of Parry [11], it suffices to confirm that the above words are lexicographically less (or equal) than itself at any starting point. $\square$

This lemma assures that if β is a Pisot unit with (F), then $-1 \le b \le a+1$ and $a \ge 0$. Thus our remaining task is to show the converse. By using Theorem 1, it suffices to show property (F) for the following three cases:

$$x^3 - ax^2 + x - 1 \quad a \ge 2,$$

$$x^3 - ax^2 - 1 \quad a \ge 1$$

and

$$x^3 - ax^2 - (a+1)x - 1 \quad a \ge 0.$$

First two cases are treated in the next section. (Note that the second case can be shown by the result of [7] as well.) The third case is most difficult and will be treated in section 4. It should be noted here, Theorem 3 can be rephrased as

Theorem 4. *A cubic Pisot unit β has property (F) if and only if $d(1,\beta)$ is finite.*

Proof. When $c = -1$, there exist a positive conjugate other than β. In this case, $d(1,\beta)$ can not be finite, by the proof of Proposition 1 in [1]. When $c = 1$, Lemma 2 shows the assertion, if we assume the truth of Theorem 3. □

The statement of Theorem 4 is not true when we consider quaratic Pisot units. See example 2 of [6]. Here we state another implication of Lemma 2.

Theorem 5. *Let β be a cubic Pisot unit with (F). Then each tile T_ω corresponding to β is arcwise connected.*

Proof. By using Theorem 3 in [3], the tile $T_.$ is arcwise connected if the last letter of $d(1,\beta)$ is 1. Thus Lemma 2 implies the arcwise connectedness of $T_.$. The proof for other tiles T_ω are similar. □

In [3], it is conjectured that the last letter of $d(1,\beta)$ for a Pisot unit with (F) is always 1.

3. Finiteness for $x^3 - ax^2 + x - 1$ and $x^3 - ax^2 - 1$

Put $f_1(x) = x^3 - ax^2 + x - 1$ with $a \ge 2$ and $f_2(x) = x^3 - ax^2 - 1$ with $a \ge 1$. This section, as a whole, is devoted to a proof of the property (F) for Pisot numbers β whose irreducible polynomial is f_i $(i = 1, 2)$. Denote by $D(f)$ the discriminant of $f \in \mathbb{Z}[x]$. Then we have $D(f_1) = -4a^3 + a^2 + 18a - 31$ and $D(f_2) = -4a^3 - 27$. So the Pisot numbers β defined by f_i $(i = 1, 2)$ are not totally real, since $D(f_i) < 0$. Let β' be a fixed complex conjugate of β and $\beta'' = \overline{\beta'}$. Designate x' and x'' for the corresponding conjugates of $x \in \mathbb{Q}(\beta)$. When $a \le 9$, the assertion of Theorem 3 is proved by the direct application of Theorem 2. Thus we will show, the property (F) holds for the Pisot numbers corresponding to f_i $(i = 1, 2)$ when $a \ge 10$. To prove this, we modify slightly the statement of Theorem 2. Since $x \in \mathrm{Fin}(\beta)$ is equivalent to $\beta x \in \mathrm{Fin}(\beta)$, it suffices to show that each element of

$$\left\{ x \in \mathbb{Z}[\beta] \;\middle|\; 0 < x \le \beta^m, |x'| \le \frac{[\beta]|\beta'|^m}{1 - |\beta'|} \right\}$$

has finite beta expansion in base β with any fixed integer m. Hereafter we put $m = 2$. Each $x \in \mathbb{Z}[\beta]$ has a form $x = x_0 + x_1\beta + x_2\beta^2$ with integers x_i $(i = 0, 1, 2)$. Thus we shall prove finiteness for the numbers $x_0 + x_1\beta + x_2\beta^2$ with

$$0 < x_0 + x_1\beta + x_2\beta^2 \quad \le \quad \beta^2, \tag{2}$$

$$|x_0 + x_1\beta' + x_2(\beta')^2| \quad \le \quad \frac{[\beta]|\beta'|^2}{1 - |\beta'|} \le \frac{1}{1 - \sqrt{\beta}}. \tag{3}$$

Writing these inequalities in a matrix form, we have

$$(x_0, x_1, x_2)\begin{pmatrix} 1 & 1 & 1 \\ \beta & \beta' & \beta'' \\ \beta^2 & (\beta')^2 & (\beta'')^2 \end{pmatrix} = (u_0, u_1, u_2),$$

with $u_0 \le \beta^2$ and $|u_i| \le 1/(1-\sqrt{\beta})$ $(i = 1, 2)$. Multiplying the inverse matrix,

$$(x_0, x_1, x_2) = \frac{(u_0, u_1, u_2)}{\sqrt{|D(f_i)|}}\begin{pmatrix} \beta'\beta''(\beta''-\beta') & -(\beta'')^2+(\beta')^2 & \beta''-\beta' \\ -\beta\beta''(\beta''-\beta) & (\beta'')^2-\beta^2 & -\beta''+\beta \\ \beta\beta'(\beta'-\beta) & -(\beta')^2+\beta^2 & \beta'-\beta \end{pmatrix}$$

with $i = 1, 2$. Thus we have the estimate

$$\begin{aligned} |x_0| &\le \frac{1}{\sqrt{|D(f_i)|}}\left\{\beta^2|\beta'\beta''(\beta''-\beta')| + \frac{2}{1-\beta^{-1/2}}|\beta\beta''(\beta''-\beta)|\right\} \\ &\le \frac{1}{\sqrt{|D(f_i)|}}\left\{2\sqrt{\beta} + \frac{2(1+\beta)\sqrt{\beta}}{1-\beta^{-1/2}}\right\}, \end{aligned}$$

with $i = 1, 2$. The right hand side is $1 + O(a^{-1/2})$. By a more precise estimate, this inequality yields $|x_0| \le 1$ when $a \ge 8$. For x_2, we have

$$\begin{aligned} |x_2| &\le \frac{1}{\sqrt{|D(f_i)|}}\left\{\beta^2|\beta''-\beta'| + \frac{2}{1-\beta^{-1/2}}|\beta''-\beta|\right\} \\ &\le \frac{1}{\sqrt{|D(f_i)|}}\left\{2\beta\sqrt{\beta} + \frac{2(1+\beta)}{1-\beta^{-1/2}}\right\}. \end{aligned}$$

In the same manner, this estimate implies $|x_2| \le 1$ when $a \ge 6$. Thus it suffices to show the finiteness when $|x_0| \le 1$ and $|x_2| \le 1$. We shall prove the assertion by classifying into three case according to the value of x_2.

Case $x_2 = -1$. By the inequality (2), $\beta - \beta^{-1} < x_1 < 2\beta + \beta^{-1}$. Thus we have

$$\frac{1}{1-\beta^{-1/2}} > |x_0 + x_1\beta' - (\beta')^2| > \left(\beta - \frac{1}{\beta}\right)\beta^{-1/2} - 1 - \frac{1}{\beta},$$

by (3). This inequality holds only when $\beta \le 7.57$. This implies that $a \le 7$ which contradicts with the assumption.

Case $x_2 = 0$. By (2), $-1/\beta \le x_1 \le \beta + 1/\beta$. Since x_1 is an integer, we have $0 \le x_1 < [\beta] + 1$. Put $x_0 = 0$. When $x_1 \le [\beta]$, $x_0 + x_1\beta + x_2\beta^2 = x_1\beta$ itself is a beta expansion. If $x_1 = [\beta]+1$. Then by (3), we have $1/(1-\beta^{-1/2}) \ge |x_0+x_1\beta'| \ge \sqrt{\beta} - 1$. This inequality can not be true when $a \ge 8$. Next we assume $x_0 = 1$. If $x_1 \le [\beta] - 1$, then $x_0 + x_1\beta + x_2\beta^2 = 1 + x_1\beta$ itself satisfies (1). When $x_1 \ge [\beta]$, we have $1/(1-\beta^{-1/2}) \ge |x_0 + x_1\beta'| \ge [\beta]/\sqrt{\beta} - 1 \ge \sqrt{\beta} - 1/\sqrt{\beta} - 1$. This can not be the case when $a \ge 10$. Assume $x_0 = -1$. Then (2) implies $x_1 \ge 1$. Thus we can carry down the digit by $d(1, \beta)$ in Lemma 2:

$$-1 + x_1\beta = x_1, (-1) = \begin{cases} (x_1 - 1), ([\beta]-1).[\beta], 0, 1 & \text{if } i = 1 \\ (x_1 - 1), ([\beta]-1).0, 1 & \text{if } i = 2, \end{cases}$$

to obtain finite beta expansion. Thus we have completed the proof for $x_2 = 0$.

Case $x_2 = 1$. Similarly we see, $-\beta - x_0/\beta \le x_1 \le 1/\beta$. Since x_1 is an integer, we have $-\beta - x_0/\beta \le x_1 \le 0$. First, suppose $x_0 = 0$. We only have to consider $-[\beta] \le x_1 \ne 0$. Then we have

$$x_1\beta + \beta^2 = 1, x_1, 0. = \begin{cases} ([\beta]+x_1), [\beta].0, 1 & \text{if } i = 1 \\ ([\beta]+x_1), 0.1 & \text{if } i = 2 \end{cases}$$

by $d(1,\beta)$ in Lemma 2. Condition (1) is fulfilled in either case. Next we assume $x_0 = 1$. Hence $-[\beta] - 1 \le x_1 \le 0$. If $x_1 = 0$ then $x_0 + x_1\beta + x_2\beta^2 = 1 + \beta^2$ itself is a beta expansion. When $-[\beta] - 1 \le x_1 \le -1$, we see

$$1 + x_1\beta + \beta^2 = 1, x_1, 1. = \begin{cases} ([\beta]+x_1+1), 0, 1 & \text{if } i = 1 \\ ([\beta]+x_1), 0.1 & \text{if } i = 2. \end{cases}$$

Then we see these are the desired finite beta expansions. Herein we used the irreducible polynomial $x^3 = ([\beta]+1)x^2 - x + 1$ instead of $d(1,\beta)$ in the above case $i = 1$. The case when $i = 2$ and $x_1 = -[\beta] - 1$ can be abandoned, because the right hand side is negative.

Lastly we consider the case when $x_0 = -1$. Now we see $-[\beta] \le x_1 \le 0$. Similarly as above,

$$1, x_1, (-1). = \begin{cases} ([\beta]+x_1), ([\beta]-1).0, 1 & \text{if } i = 1 \\ ([\beta]+x_1), (-1).1 = ([\beta]+x_1-1), ([\beta]-1).1, 1 & \text{if } i = 2, \end{cases}$$

which shows the assertion. Here when $i = 2$ and $x_1 = -[\beta]$, the value of the right hand side is negative, which can be omitted. The proof of the property (F) for polynomials f_i $(i = 1, 2)$ is now completed.

4. Finiteness for $x^3 - ax^2 - (a+1)x - 1$

To complete our proof of Theorem 3, we shall treat the final case $f(x) = x^3 - ax^2 - (a+1)x - 1$ for $a \ge 0$ in this section. We will prove property (F) for these polynomials with $a \ge 15$, while other cases are shown by Theorem 2. The discriminant $D(f)$ is $a^4 + 2a^3 - 5a^2 - 6a - 23$ which is positive for $a \ge 3$. Thus we have to treat totally real cases. Let β' and β'' be the conjugates of β with $-1 < \beta' < \beta'' < 0 < 1 < \beta$. We need precise asymptotic behaviors of these conjugates when $a \to \infty$ (or $\beta \to \infty$):

Lemma 3. *If* $a \geq 8$*,*

$$\begin{aligned} 2 &\leq (\beta' + 1 - \frac{1}{\beta})\beta^3 \leq 2.5 \\ -1.5 &\leq (\beta'' + \frac{1}{\beta} + \frac{1}{\beta^2})\beta^3 \leq -1 \end{aligned} \tag{4}$$

Proof. Since $\beta^3 - a\beta^2 - (a+1)\beta - 1 = 0$, we have the expansion

$$a = \beta - 1 - \frac{1}{\beta^2} + \frac{1}{\beta^3} - \frac{1}{\beta^4} + \frac{1}{\beta^5} + \dots. \tag{5}$$

On the other hand, as $\beta' + \beta'' = a - \beta$ and $\beta'\beta'' = 1/\beta$, we have $\beta' = (a - \beta - \sqrt{(a-\beta)^2 - 4/\beta})/2$ and $\beta'' = (a - \beta + \sqrt{(a-\beta)^2 - 4/\beta})/2$. Putting the truncated inequality of (5):

$$\beta - 1 - \frac{1}{\beta^2} + \frac{4}{5\beta^3} \leq a \leq \beta - 1 - \frac{1}{\beta^2} + \frac{1}{\beta^3}$$

into these expressions, we can derive the desired estimations. □

Now we use Theorem 2 in a modified form as in the previous section. Then it suffices to show the finiteness for the elements $x_0 + x_1\beta + x_2\beta^2$ which satisfy:

$$\begin{aligned} 0 < x_0 + x_1\beta + x_2\beta^2 &< \beta^2 \\ |x_0 + x_1\beta' + x_2(\beta')^2| &< \frac{[\beta]|\beta'|^2}{1 - |\beta'|} \\ |x_0 + x_1\beta'' + x_2(\beta'')^2| &< \frac{[\beta]|\beta''|^2}{1 - |\beta''|}. \end{aligned} \tag{6}$$

Applying Lemma 3, we see

$$|x_0 + x_1\beta' + x_2(\beta')^2| < \beta^2 \tag{7}$$

and

$$|x_0 + x_1\beta'' + x_2(\beta'')^2| < \frac{1}{\beta}\left(1 + \frac{4}{\beta}\right) \quad \text{for } \beta \geq 8. \tag{8}$$

Hereafter we shall prove the required finiteness for the solutions (x_0, x_1, x_2) of these three inequalities (6),(7) and (8). Now we show

Lemma 4. *If* $a \geq 7$*, then* $|x_0 + x_2| \leq 2$*.*

Proof. Putting the estimates (6) and (8) together,

$$\left| \left(\frac{\beta}{\beta''} - 1\right) x_0 + \left(\beta\beta'' - \beta^2\right) x_2 \right| \leq \beta^2 + \frac{1}{|\beta''|}\left(1 + \frac{4}{\beta}\right). \tag{9}$$

Now we derive estimates of x_0, by the method of the previous section. Then we have, for $\beta \geq 8$,

$$(x_0, x_1, x_2) = \frac{(u_0, u_1, u_2)}{\sqrt{D(f)}} \begin{pmatrix} \beta'\beta''(\beta''-\beta') & -(\beta'')^2+(\beta')^2 & \beta''-\beta' \\ -\beta\beta''(\beta''-\beta) & (\beta'')^2-\beta^2 & -\beta''+\beta \\ \beta\beta'(\beta'-\beta) & -(\beta')^2+\beta^2 & \beta'-\beta \end{pmatrix},$$

with $|u_0| \leq \beta^2, |u_1| \leq \beta^2$ and $|u_2| \leq \beta^{-1}(1+4/\beta)$. Applying Lemma 3, we can show

$$|x_0| \leq \frac{\beta^3+\beta^2+4.5\beta+7}{\sqrt{D(f)}}.$$

By using (5), this implies

$$|x_0| \leq \beta + 3 \quad \text{for } a \geq 7. \tag{10}$$

Now, by (9),

$$\left|\beta\beta''-\beta^2\right| |x_0+x_2| \leq |x_0| \left|\beta\beta''-\beta^2-\frac{\beta}{\beta''}+1\right| + \frac{1}{|\beta''|}\left(1+\frac{4}{\beta}\right)+\beta^2. \tag{11}$$

By using Lemma 3, we see

$$\left|\beta\beta''-\beta^2\right| \geq 1+\beta^2, \tag{12}$$

$$\left|\beta\beta''-\beta^2-\frac{\beta}{\beta''}+1\right| \leq 1+\beta, \tag{13}$$

and

$$\frac{1}{|\beta''|}\left(1+\frac{4}{\beta}\right)+\beta^2 \leq \beta^2+\beta+4. \tag{14}$$

Combining (10), (11),(12),(13) and (14), we have

$$|x_0+x_2| \leq 2 + \frac{5\beta+5}{\beta^2+1},$$

which shows $|x_0+x_2| \leq 2$ for $\beta \geq 8$. □

We will prove the key lemma.

Lemma 5. *Let (ξ_0, ξ_1, ξ_2) and (η_0, η_1, η_2) be the solutions of three inequalities (6),(7) and (8) with $\xi_0 = \eta_0$. Then we have $|\xi_1 - \eta_1| \leq 2$ and $|\xi_2 - \eta_2| \leq 1$ for $a \geq 12$.*

Proof. By (8),

$$\left|(\xi_1-\eta_1)\beta''+(\xi_2-\eta_2)(\beta'')^2\right| \leq \frac{2}{\beta}\left(1+\frac{4}{\beta}\right)$$

Since $\xi_0 = \eta_0$, using Lemma 4, we have $|\xi_2 - \eta_2| \le 4$. Now we have, by Lemma 3,

$$|(\xi_1 - \eta_1)\beta''| \le \frac{2}{\beta} + \frac{13}{\beta^2},$$

for $\beta \ge 10$. Again by Lemma 3,

$$|\xi_1 - \eta_1| < 2 + \frac{12}{\beta}.$$

Thus we see $|\xi_1 - \eta_1| \le 2$ for $\beta \ge 13$. Now (6) implies

$$|(\xi_1 - \eta_1)\beta + (\xi_2 - \eta_2)\beta^2| \le \beta^2.$$

Thus we have $|\xi_2 - \eta_2| \le 1 + 2/\beta$. As the left hand side is an integer, we see the assertion. □

Lemma 5 provides us with a way to find out all solutions of inequalities (6),(7) and (8), by constructing a certain special kind of solutions. Actually, we can construct a series of solutions, denoted by *fundamental solutions*, by the next lemma.

Lemma 6. *The elements* $m(a + \beta^{-1})$ *for* $m = 1, 2, \dots, [\beta]$ *and* $\beta^2 + m(a + \beta^{-1})$ *for* $m = -1, -2, \dots, -[\beta]$ *satisfy the desired three inequalities (6),(7) and (8) for* $a \ge 9$

For the simplicity of notations, we use the term 'fundamental solutions' to express above $2[\beta]$ elements, although we are on the way to prove it.

Proof. First note $a + \beta^{-1}$ is also a unit. In fact, we have

$$-(x - a)^3 f(1/(x - a)) = x^3 + (1 - 2a)x^2 + (a^2 - a)x - 1.$$

The inequality (6) for fundamental solutions is obviously fulfilled. By (4), we see

$$|a + (\beta')^{-1}| \le a - 1,$$

which implies (7) for fundamental solutions. Since

$$a + (\beta'')^{-1} = (a + \beta^{-1})^{-1}(a + (\beta')^{-1})^{-1},$$

we can easily get the estimate

$$|a + (\beta'')^{-1}| \le \frac{1}{\beta^2}\left(1 + \frac{4}{\beta}\right),$$

for $a \ge 9$, by using Lemma 3. This estimation is enough to show (8) for fundamental solutions. □

These fundamental solutions have concrete and beautiful finite beta expansions in base β.

Lemma 7. *We have*

$$k(a+\beta^{-1}) = -\beta + \sum_{i=0}^{k-1}(k-i)\beta^{-2i+1} + \sum_{i=1}^{k}(a-k+i)\beta^{-2i+2} + \beta^{-2k+1}$$
$$= (k-1)(a-k+1).(k-1)(a-k+2)(k-2)(a-k+3)\dots,2,(a-1),1,a,1$$

and

$$\beta^2 - k(a+\beta^{-1}) =$$
$$(a-k+1)(k-1).(a-k+2)(k-2)(a-k+3)\dots,3,(a-2),2,(a-1),1,a,1$$

for $k = 1, 2, \dots, [\beta]$. *Right hand sides of these expansions satisfies (1).*

Proof. Note for any k,

$$k(a+\beta^{-1}) = (k-1)\beta + (a-k+1) + \beta^{-1} + (k-1)(a+\beta^{-1})\beta^{-2}.$$

Using this recursively, we get the first formula. One can show the second one, by using

$$\beta^{-1}(\beta^2 - k(a+\beta^{-1})) + (k-1)\beta = k(a+\beta^{-1}).$$

□

The reader might feel a little bit curious on a sudden appearance of $a+\beta^{-1}$. Let $\epsilon = (a+\beta^{-1})^{-1}$. Then we see $|\epsilon| < 1, |\epsilon'| < 1$ and $|\epsilon''| > 1$. Hence β and $a+\beta^{-1}$ are independent units of our totally real field $\mathbb{Q}(\beta)$. Actually, one can show by Theorem 2 of K.Minemura [10], they form a system of fundamental units of this field, when $D(f)$ is positive and square free. Note that we need slight change of notations like $1/\theta = 1 + 1/\beta$. R.Okazaki kindly informed me of this fact. Now we can proceed into the last step.

Proof of finiteness for $x^3 - ax^2 - (a+1)x - 1$. We need a precise variant of (10). In the notation of the proof of Lemma 4, we have

$$\begin{aligned} |u_1| &\le \beta^2 \frac{(1-1/\beta+2.5/\beta^2)^2}{1-2/\beta^2} \\ &\le \beta^2(1-2/\beta+8/\beta^2). \end{aligned}$$

Thus we have, by Lemma 3,

$$|x_0| \le \frac{\beta^3-\beta^2+10.5\beta+12}{\sqrt{D(f)}},$$

for $\beta \ge 10$. By using (5), this implies

$$|x_0| \le \beta \quad \text{for } a \ge 15. \tag{15}$$

Noting $a+\beta^{-1} = \beta^2 - a\beta - 1$ and combining (15), Lemma 5 and Lemma 6, it suffices to show the finiteness for

$$x_0(a+\beta^{-1}) + \kappa_1\beta + \kappa_2\beta^2,$$

with $0 \leq x_0 \leq \beta$ and

$$\beta^2 + x_0(a + \beta^{-1}) + \kappa_1\beta + \kappa_2\beta^2,$$

with $-1 \geq x_0 \geq -\beta$. Here κ_i $(i = 1, 2)$ are integers with $|\kappa_1| \leq 2, |\kappa_2| \leq 1$. We classify the proof in several cases.

Case $1 \leq x_0 \leq [\beta] - 2$. We only have to show the finiteness when the value

$$x_0(a + \beta^{-1}) + \kappa_1\beta + \kappa_2\beta^2,$$

is positive. Thus we see $\kappa_2 \geq 0$. In fact, by Lemma 7 and Lemma 2, $x_0(a+\beta^{-1})+\kappa_1\beta \leq (x_0+1)\beta+(a-x_0+1)+\dots$ is clearly less than β^2 when $x_0+1 \leq a = [\beta]-1$. First, let $\kappa_2 = 0$. When $\kappa_1 = 0, 1, 2$ then $x_0(a + \beta^{-1}) + \kappa_1\beta = (x_0 - 1 + \kappa_1)\beta + (a - x_0 + 1) + \dots$ is a beta expansion. If κ_1 is negative, then the same expression satisfies (1) unless $x_0(a + \beta^{-1}) + \kappa_1\beta$ is negative.

Second, let $\kappa_2 = 1$. By using (6), we see $\kappa_1 = -1$ or -2 and $x_0 \leq 2$. But $x_0 = 2$ implies $\kappa_1 = -2$ and $|2(a + (\beta'')^{-1}) - 2\beta'' + (\beta'')^2| > (1 + 4/\beta)/\beta$ which contradicts with (8). When $x_0 = 1$. Then

$$a + \beta^{-1} + \kappa_1\beta + \beta^2 = (a + 1 + \kappa_1)\beta + a + \beta^{-1} + a\beta^{-2} + \beta^{-3}$$

is a finite beta expansion.

Case $x_0 = [\beta]-1 = a$. When $\kappa_2 > 0$, $a(a+\beta^{-1})+\kappa_1\beta+\kappa_2\beta^2$ does not satisfy (6). First, let $\kappa_2 = 0$. Then

$$a(a + \beta^{-1}) + \kappa_1\beta = (a - 1 + \kappa_1)\beta + 1 + (a - 1)\beta^{-1} + \dots$$

is a beta expansion when $\kappa_1 \leq 1$. When $\kappa_1 = 2$, it is greater than β^2 which does not satisfy (6).

Second, let $\kappa_2 = -1$. When $\kappa_1 < 2$, the value $a(a+\beta^{-1})+\kappa_1\beta-\beta^2$ is negative. Thus we only have to consider the case $\kappa_1 = 2$. Noting the identity:

$$a(a + \beta^{-1}) + 2\beta - \beta^2 = 1 + \beta^{-2}((a - 2)(a + \beta^{-1}) + \beta),$$

we see the right hand side has finite beta expansion, by Lemma 7.

Case $x_0 = [\beta] = a + 1$. We can show this case almost similarly. By (6), we have $\kappa_2 \leq 0$. First, let $\kappa_2 = 0$. Then we have $\kappa_1 \leq 0$, by (6). Then

$$(a + 1)(a + \beta^{-1}) + \kappa_1\beta$$

has a finite expansion by Lemma 7. Second, let $\kappa_2 = -1$. This implies $\kappa_1 > 0$ by (6). When $\kappa_1 = 1$, we have an identity:

$$(a + 1)(a + \beta^{-1}) + \beta - \beta^2 = \beta^{-2}((a - 1)(a + \beta^{-1}) + \beta),$$

which shows the finiteness by Lemma 7. Adding β to both hand sides we get the finiteness for $\kappa_1 = 2$.

Case $x_0 = 0$. We see $\kappa_2 \geq 0$. One can check the desired finiteness very easily.

Now we treat the case when $x_0 < 0$. The proof is almost parallel to the case when $x_0 > 0$. Thus we will omit the details.

Case $-1 \geq x_0 \geq -[\beta] + 2 = -a + 1$. One see $\kappa_2 \leq 0$ by (6). First, let $\kappa_2 = 0$. Then we see, by Lemma 7, $\beta^2 + x_0(a+\beta^{-1}) + \kappa_1\beta$ has finite beta expansion unless it is greater than β^2. Second, let $\kappa_2 = -1$. Then we see $\kappa_1 > 0$ and $x_0 = -1$ or -2. But when $\kappa_1 = 2$, there are no cases which satisfy (8). Thus we have to consider $\kappa_1 = 1$ and $x_0 = -1$. Then $\beta - (a + \beta^{-1}) = a\beta^{-1} + \beta^{-2}$ is a beta expansion.

Case $x_0 = -[\beta] - 1 = -a$. We have $\kappa_2 \geq 0$. First, let $\kappa_2 = 0$. Then we see the expansion generated by Lemma 7 satisfies (1) when $\kappa_1 \geq -1$. The value $\beta^2 - a(a + \beta^{-1}) + \kappa_1\beta$ is negative when $\kappa_1 = -2$. Second, let $\kappa_2 = 1$. Then we only have to show when $\kappa_1 = -2$. Considering the identity:

$$2\beta^2 - a(a + \beta^{-1}) - 2\beta = a\beta + (a - 1) + \beta^{-2}(\beta^2 - (a - 2)(a + \beta^{-1})),$$

we get the assertion.

Case $x_0 = -[\beta] = -a-1$. We see $\kappa_2 \geq 0$. First, let $\kappa_2 = 0$. When $\kappa_1 < 0$, the value $\beta^2 - (a+1)(a+\beta^{-1}) + \kappa_1\beta$ is negative. If $\kappa_1 \geq 0$, it has finite beta expansion by Lemma 7. Second, let $\kappa_2 = 1$. Then we have $\kappa_1 < 0$. When $\kappa_1 = -1$, then the identity:

$$2\beta^2 - (a + 1)(a + \beta^{-1}) - \beta = a\beta + a + \beta^{-2}(\beta^2 - (a - 1)(a + \beta^{-1}))$$

assures the finiteness. Subtracting β from both sides, we get the assertion for $\kappa_1 = -2$. □

Thus we have completed the proof of Theorem 3.

5. Proofs of Propositions

Proof of Proposition 1. In [6], it is shown, if $\mathbb{Z}_{\geq 0} \subset \mathrm{Fin}(\beta)$ then β is a Pisot number or a Salem number. Thus our task is to show that the later case is absurd. Suppose that β is a Salem number and $\mathbb{Z}_{\geq 0} \subset \mathrm{Fin}(\beta)$. In [15], it is proved that β is a root of the reciprocal polynomial. This shows β has just one conjugate $1/\beta$ in the interior of the unit circle and $\deg \beta \geq 4$. Let k be a positive integer and consider the beta expansion:

$$1 + [\beta^k] = \beta^k + \sum_{i=1}^{q} a_{-i}\beta^{-i},$$

with $a_{-q} \neq 0$. Let η be a conjugate of β with $|\eta| = 1$. Taking conjugate of both sides,

$$1 + [\beta^k] = \eta^k + \sum_{i=1}^{q} a_{-i}\eta^{-i}.$$

Considering the absolute value, we see

$$1 + [\beta^k] \leq 1 + \sum_{i=1}^{q} a_{-i} \leq 1 + q[\beta].$$

On the other hand, by the conjugate map which send β to $1/\beta$,

$$1 + [\beta^k] = \beta^{-k} + \sum_{i=1}^{q} a_{-i}\beta^i > \beta^q.$$

This implies $k \geq q$. Summing up, we have $[\beta^k]/[\beta] \leq k$. This inequality can not be true when k is large. □

To prove Proposition 2, we need to a result on the tiling for the Pisot units which are not necessary assumed to have property (F). Note $\mathbb{Z}[\beta^{-1}] = \mathbb{Z}[\beta]$, since β is a unit.

Let Fr be the set of words corresponding to the fractional parts of elements of $\mathbb{Z}[\beta]_{\geq 0}$. By the result of [16] and [5], the fractional parts appeared in Fr are eventually periodic. Moreover the fractional parts are eventually classified into finite types. Recall that $\mathbb{Z}[\beta]_{\geq 0}$ is dense in $\mathbb{R}^{m-1}$ by Proposition 1 of [3]. Thus we have

$$\mathbb{R}^{m-1} = \bigsqcup_{\omega \in \mathrm{Fr}} \overline{\Phi(S_\omega)}.$$

It can be shown, for any element $\omega \in$ Fr, $T_\omega = \overline{\mathrm{Inn}(T_\omega)}$ and $\partial(T_\omega)$ is nowhere dense in $\mathbb{R}^{m-1}$, which is a generalization of the results of [3]. Further the $m-1$ dimensional Lebesgue measure of the $\partial(T_\omega)$ is 0, which will be shown in [4].

Proof of Proposition 2. Suppose that the origin is an inner point of $T_.$ and $x \in \mathbb{Z}[\beta^{-1}]$ has infinite beta expansion. Consider the sequence $\beta^k x \quad (k = 0, 1, \dots)$. By the definition of the beta expansion, we see each $\beta^k x$ has infinite beta expansion as well. But the sequence $\Phi(\beta^k x) \quad (k = 0, 1, \dots)$ converges to the origin as β is a Pisot number. Since $\beta^k x$ is an accumulation point of $\mathrm{Inn}(T_\omega)$ with $\omega \neq' .'$, the origin can not be an inner point of $T_.$ Here we used the fact that the boundary of the tile is nowhere dense in $\mathbb{R}^{m-1}$ □

References

[1] S.Akiyama, Pisot numbers and greedy algorithm, Number Theory, Diophantine, Computational and Algebraic Aspects, Edited by K. Györy, A. Pethö and V.T.Sós, 9–21, de Gruyter 1998.

[2] S.Akiyama and T.Sadahiro, A self-similar tiling generated by the minimal Pisot number, Acta Math. Info. Univ. Ostraviensis 6 (1998) 9–26, (Proceedings of 13th Czech and Slovak Conference on Number Theory)

[3] S.Akiyama, Self affine tiling and Pisot numeration system, to appear in 'Number Theory and its Applications', ed. by K. Györy and S. Kanemitsu, Kluwer

[4] S.Akiyama, On the boundary of self affine tilings generated by Pisot numbers, in preparation.

[5] A.Bertrand, Dévelopment en base de Pisot et répartition modulo 1, C.R.Acad.Sc., Paris 385 (1977) 419–421

[6] C.Frougny and B.Solomyak, Finite beta-expansions, Ergod. Th. and Dynam. Sys. 12 (1992) 713–723

[7] M.Hollander, Linear numeration systems, finite beta expansions, and discrete spectrum of substitution dynamical systems, thesis of Washington Univ.

[8] S.Ito and M.Kimura, On Rauzy Fractal, Japan J. Indust. Appl. Math. 8 (1991) 461-486

[9] R.Kenyon, The construction of self-similar tilings, Geom. Funct. Anal. 6 (1996), no. 3, 471–488.

[10] K.Minemura, On totally real cubic fields whose unit group is of type $\{\theta + r, \theta + s\}$, preprint.

[11] W.Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungar. 11 (1960) 269-278

[12] B.Praggastis, Markov partition for hyperbolic toral automorphism, Ph.D.Thesis, Univ. of Washington, 1992.

[13] G.Rauzy, Nombres Algébriques et substitutions, Bull. Soc. France 110 (1982) 147-178.

[14] A.Rényi, Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungar 8 (1957) 477–493.

[15] R.Salem, Algebraic Numbers and Fourier Analysis, D.C.Gealth & Co., Boston, 1963

[16] K.Schmidt, On periodic expansions of Pisot numbers and Salem numbers, Bull. London Math. Soc. 12 (1980) 269–278

[17] B.Solomyak, Dynamics of self-similar tilings. Ergodic Theory Dynam. Systems 17 (1997), no. 3, 695–738.

[18] W.P.Thurston, Groups, Tilings and Finite state automata, AMS Colloquium lectures, 1989.