ALGORITHM FOR DETERMINING PURE POINTEDNESS OF SELF-AFFINE TILINGS

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Abstract. Overlap coincidence in a self-affine tiling in $\mathbb{R}^d$ is equivalent to pure point dynamical spectrum of the tiling dynamical system. We interpret the overlap coincidence in the setting of substitution Delone set in $\mathbb{R}^d$ and find an efficient algorithm to check the pure point dynamical spectrum. This algorithm is easy to implement into a computer program. We give the program and apply it to several examples. In the course of the proof of the algorithm, we show a variant of the conjecture of Urbański (Solomyak [43]) on the Hausdorff dimension of the boundaries of fractal tiles.

Keywords: Pure point spectrum, Self-affine tilings, Coincidence, Substitution Delone sets, Meyer sets, Algorithm, Quasicrystals, Hausdorff dimension, Fractals.

1. Introduction

To model self-inducing structures of dynamical systems, symbolic dynamical systems associated with substitutions play an important role and many works describe their spectral properties and geometric realizations (see [34]). To extend the symbolic substitutive systems to higher dimensions, self-affine tiling dynamical systems are studied in detail in [41, 26] and many related studies are done along this line. These tiling dynamical systems share many properties with the symbolic substitutive systems and are intimately related to the explicit construction of Markov partitions. It is a subtle question to determine whether a given tiling dynamical system has pure point dynamical spectrum or not. It is known from [41, 26] that 'overlap coincidence' (see Def. 2.4) is an equivalent criterion to check this. However the overlap coincidence was not easy to compute there in practice because it requires topological properties of the tiles. To settle this difficulty, we shall employ the duality between self-affine tilings and substitutive Delone sets [22, 26, 23]. An aim of this paper is to transfer the overlap coincidence to substitution Delone sets, find a computable algorithm to check the pure pointedness and implement it into a program language.

Further motivation to show the pure pointedness comes from the study of aperiodic order. It is an interesting question to ask what kind of point sets, modeling atomic configurations, present pure point diffraction. This is related with the understanding of the fundamental structures of quasicrystals. It has been known from [25, 14, 4] that pure point diffraction

Date: September 28, 2010.

The first author is supported by the Japanese Society for the Promotion of Science (JSPS), grant in aid 21540010. The second author is grateful for the support of KIAS in this research and the support of Basic Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2010-0011150).

spectrum is equivalent to pure point dynamical spectrum in quite a general setting. So the algorithm we give here can be used for checking pure point diffraction of general self-affine quasi-periodic structures.

There are many equivalent criteria to the pure point dynamical spectrum in literature. Among them, coincidences are very well known as a characterization of the pure point dynamical spectrum. There are many different notions of coincidences but basically they imply the same thing. In 1-dim substitution sequences, Dekking's coincidence is well-known for the case of constant-length substitutions [11]. For 1-dim irreducible Pisot substitution sequences or tilings, super coincidence, strong coincidence, geometric coincidence, balanced pairs, and boundary graph are known [16, 2, 6, 40, 38, 34]. In higher dimensions, modular coincidence was introduced for lattice substitution Delone sets [24, 26, 12], and overlap coincidence and algebraic coincidence are known for substitution tilings and substitution Delone sets under the assumption of Meyer property [41, 23]. We are going to use the overlap coincidence for computation here.

We note that it is essential to assume the Meyer property of the corresponding substitution Delone set. Otherwise, the algorithm will either not terminate as the number of overlaps becomes infinite, or terminate with incorrect outputs. It is shown in [27] that substitution Delone sets with pure point dynamical spectrum necessarily have the Meyer property. It is also studied in [28] under which conditions on the expansion maps of the substitutions, the point sets are guaranteed to have the Meyer property.

There are a few results in literature for the actual computation of coincidence. For 1-dimension unit Pisot substitutions and self-affine tilings coming from their geometric realizations, computable algorithm is discussed in [37, 38] using the boundary graph. For irreducible 1-dimension Pisot substitution, balanced pair algorithm is implemented in [40]. For higher dimensions, Dekking's coincidence and modular coincidence are used for the case of lattice substitution Delone sets in $\mathbb{R}^d$ [11, 24, 26]. It was shown in [12] that the modular coincidence in lattice substitution Delone sets can be determined within some bounded iterations. The given bound is exponential to the number of colours of the Delone sets. It was conjectured in [12] that the lowest upper bound is a quadratic value of m. Here we shall give a quadratic bound of m for overlap coincidence. Note that checking overlap coincidence takes less time than modular coincidence, though two coincidences are equivalent (see Remark 1). Moreover overlap coincidence can be used not only for the lattice substitution Delone sets but also for the substitution Delone sets with the Meyer property.

In this paper we compute overlap coincidence for general self-affine tilings. Our method covers, non-unit cases, higher dimensional and non-lattice based self-affine tilings. With regard to computation, already in the original paper by Solomyak [41], the number of overlaps becomes too large to handle by hand. Apart from 1-dimensional case with connected tiles (i.e. intervals), it is quite hard to check whether translated tiles have intersection. The implementation is already difficult for polygonal tilings, and moreover, tiles often have fractal boundaries in higher dimensional cases. To overcome this difficulty, we escape from judging interior intersection. We interpret the overlaps in terms of points and translation vectors, and only care distances between the corresponding translated tiles. If the distances are within a rough bound (see (2.12)), we say they are *potential overlaps*. Of course by this change, some pairs of translated tiles may not intersect, or only meet at their boundaries. To distinguish these cases from overlaps with interior intersection, we introduce a *potential overlap graph with multiplicities*. At the expense of having a larger graph, all computation becomes simple and easy to implement into computer programs. Showing that our criterion (Theorem 4.1 (ii)) is necessary, we prove partially a variant of the conjecture which asserts that the boundaries of the self-affine tiles have Hausdorff dimension less than the space dimension d (see [43] for the conjecture).

The paper is organized in the following way: In Section 2, we give definitions and notations. As a main result, we present a mathematical algorithm computing the overlap coincidence. In Section 3 and 4, we give a justification on this algorithm. In Section 5, we have built a 'Mathematica' program implementing the algorithm and apply it to 1, 2 and 3-dimensional examples. The spectral properties of some of the examples have not been known before.

2. Preliminary

The notation and terminology we use in this paper is standard. We refer the reader to [26] for more detailed definitions and to [21] for the standard notions.

2.1. **Tilings.** We begin with a set of types (or colours) $\{1, \dots, m\}$, which we fix once and for all. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$, which is the closure of its interior, and $i = l(T) \in \{1, \dots, m\}$ is the type of T. We let $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. We say that a set P of tiles is a *patch* if the number of tiles in P is finite and the tiles of P have mutually disjoint interiors. The *support of a patch* is the union of the supports of the tiles that are in it. The *translate of a patch* P by $g \in \mathbb{R}^d$ is $g + P := \{g + T : \ T \in P\}$. We say that two patches P_1 and P_2 are *translationally equivalent* if $P_2 = g + P_1$ for some $g \in \mathbb{R}^d$. A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \bigcup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors. We always assume that any two $\mathcal{T}$-tiles with the same colour are translationally equivalent (hence there are finitely many $\mathcal{T}$-tiles up to translations). Let

$$\Xi(\mathcal{T}) := \{x \in \mathbb{R}^d : \exists\, T = (A, i), T' = (A', i) \in \mathcal{T} \text{ for } i \le m \text{ such that } A' = x + A\}.$$

We say that $\mathcal{T}$ has *finite local complexity (FLC)* if for each radius $R > 0$ there are only finitely many equivalent classes of patches whose support lies in some ball of radius R. We define $\mathcal{T} \cap A := \{T \in \mathcal{T} : \mathrm{supp}(T) \cap A \neq \emptyset\}$ for a bounded set $A \subset \mathbb{R}^d$. We say that $\mathcal{T}$ is *repetitive* if for every compact set $K \subset \mathbb{R}^d$, $\{t \in \mathbb{R}^d : \mathcal{T} \cap K = (t + \mathcal{T}) \cap K\}$ is relatively dense. We write $B_R(y)$ for the closed ball of radius R centered at y and use also B_R for $B_R(0)$.

2.2. **Delone multi-colour sets.** A *multi-colour set* or *m-multi-colour set* in $\mathbb{R}^d$ is a subset $\mathbf{\Lambda} = \Lambda_1 \times \cdots \times \Lambda_m \subset \mathbb{R}^d \times \cdots \times \mathbb{R}^d$ (m copies) where $\Lambda_i \subset \mathbb{R}^d$. We also write $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_m) = (\Lambda_i)_{i \le m}$. Recall that a Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is a *Delone multi-colour set* in $\mathbb{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \cup_{i=1}^m \Lambda_i \subset \mathbb{R}^d$ is Delone. A *cluster* of $\mathbf{\Lambda}$ is, by definition, a family $\mathbf{P} = (P_i)_{i \le m}$ where $P_i \subset \Lambda_i$ is finite for all $i \le m$. The translate of a cluster $\mathbf{P}$ by $x \in \mathbb{R}^d$ is $x + \mathbf{P} = (x + P_i)_{i \le m}$. We say that two clusters $\mathbf{P}$ and $\mathbf{P}'$ are translationally equivalent if $\mathbf{P} = x + \mathbf{P}'$ for some $x \in \mathbb{R}^d$. We say that $\Lambda \subset \mathbb{R}^d$ is a *Meyer set* if it is a Delone set and $\Lambda - \Lambda$ is uniformly discrete ([19]). We define FLC and repetitivity on Delone multi-colour sets in the same way as the corresponding properties on tilings. The types (or colours) of points on Delone multi-colour sets have the same concept as the colours of tiles on tilings.

2.3. **Substitutions.** We say that a linear map $Q : \mathbb{R}^d \to \mathbb{R}^d$ is *expansive* if all the eigenvalues of Q lie outside the closed unit disk in $\mathbb{C}$.

2.3.1. *Substitutions on tilings.*

Definition 2.1. Let $\mathcal{A} = \{T_1, \dots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of non empty patches. We say that

$\Omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a *tile-substitution* (or simply *substitution*) with an expansive map Q if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$ such that

$$\Omega(T_j) = \{u + T_i :\ u \in \mathcal{D}_{ij},\ i = 1, \dots, m\} \tag{2.1}$$

with

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le m. \tag{2.2}$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

Note that $QA_j = \text{supp}(\Omega(T_j)) = Q\text{supp}(T_j)$. The substitution (2.1) is extended to all translates of prototiles by

$$\Omega(x + T_j) = Qx + \Omega(T_j), \tag{2.3}$$

in particular,

$$\begin{aligned} \text{supp}(\Omega(x + T_j)) &= \text{supp}(Qx + \Omega(T_j)) \\ &= Qx + Q\text{supp}(T_j) \\ &= Q(x + \text{supp}(T_j)), \end{aligned} \tag{2.4}$$

and to patches and tilings by $\Omega(P) = \cup\{\Omega(T) :\ T \in P\}$. The substitution Ω can be iterated, producing larger and larger patches $\Omega^k(P)$. We say that $\mathcal{T}$ is a *substitution tiling* if $\mathcal{T}$ is a tiling and $\Omega(\mathcal{T}) = \mathcal{T}$ with some substitution Ω. In this case, we also say that $\mathcal{T}$ is a *fixed point* of Ω. We say that a substitution tiling is *primitive* if the corresponding substitution matrix S, with $S_{ij} = \sharp(\mathcal{D}_{ij})$, is primitive. A repetitive fixed point of a primitive tile-substitution with FLC is called a *self-affine tiling*. If Q is a similarity, then the tiling will be called *self-similar*. For any self affine tiling which holds (2.2), we define Φ an $m \times m$ array for which each entry is Φ_{ij},

$$\Phi_{ij} = \{f : x \mapsto Qx + d \ :\ d \in \mathcal{D}_{ij}\}$$

and call Φ a *matrix function system (MFS)* for the substitution Ω. We compose

$$\Phi \circ \Phi = ((\Phi \circ \Phi)_{ij}),$$

where $(\Phi \circ \Phi)_{ij} = \cup_{k=1}^{m} \Phi_{ik} \circ \Phi_{kj}$ and $\Phi_{ik} \circ \Phi_{kj} := \begin{cases} \{\ g \circ f \ :\ g \in \Phi_{ik}, f \in \Phi_{kj}\ \} \\ \emptyset \text{ if } \Phi_{ik} = \emptyset \text{ or } \Phi_{kj} = \emptyset\,. \end{cases}$ We write Φ^2 for $\Phi \circ \Phi$ and similarly Φ^n for n-times composition of Φ for $n \in \mathbb{Z}_+$. Let $P(\mathbb{R}^d)$ be the set of subsets of $\mathbb{R}^d$. For any $\mathbb{U} = (U_1, \cdots, U_m) \in P(\mathbb{R}^d)^m$, we write $\Phi(\mathbb{U})$ to mean $(\cup_{j \le m} \Phi_{ij}(U_j))_{i \le m}$ where $\Phi_{ij}(U_j) = \cup_{f \in \Phi_{ij}} f(U_j)$. We write $\Phi^n(x)$ for $({\Phi^n}_{ij}\{x\})_{i \le m}$ where $x \in \Lambda_j$ and $n \in \mathbb{Z}_+$.

2.3.2. *Substitutions on Delone multi-colour sets.*

Definition 2.2. $\mathbf{\Lambda} = (\Lambda_i)_{i \le m}$ is called a *substitution Delone multi-colour set* in $\mathbb{R}^d$ if $\mathbf{\Lambda}$ is a Delone multi-colour set and there exist an expansive map $Q : \mathbb{R}^d \to \mathbb{R}^d$ and finite sets $\mathcal{D}_{ij}$ for $i, j \le m$ such that

$$\Lambda_i = \bigcup_{j=1}^{m} (Q\Lambda_j + \mathcal{D}_{ij}), \quad i \le m, \tag{2.5}$$

where the unions on the right-hand side are disjoint.

We say that a cluster $\mathbf{P}$ is *legal* if it is a translate of a subcluster of a cluster generated from one point of $\mathbf{\Lambda}$, i.e. $a + \mathbf{P} \subset \Phi^k(x)$ for some $k \in \mathbb{Z}_+$, $a \in \mathbb{R}^d$ and $x \in \mathbf{\Lambda}$.

2.3.3. *Representability of $\mathbf{\Lambda}$ as a tiling.* Let $\mathbf{\Lambda}$ be a primitive substitution Delone multi-colour set. One can set up an *adjoint system of equations*

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i), \quad j \le m \tag{2.6}$$

from the equation (2.5). It is known that (2.6) always has a unique solution for which $\{A_1, \dots, A_m\}$ is a family of non-empty compact sets of $\mathbb{R}^d$. It is proved in [22, Th. 2.4 and Th. 5.5] that if $\mathbf{\Lambda}$ is a primitive substitution Delone multi-colour set, all the sets A_i from (2.6) have non-empty interiors and, moreover, each A_i is the closure of its interior. We say that $\mathbf{\Lambda}$ is *representable* (by tiles) if

$$\mathbf{\Lambda} + \mathcal{A} := \{x + T_i : \ x \in \Lambda_i, \ i \le m\}$$

is a tiling of $\mathbb{R}^d$, where $T_i = (A_i, i)$, $i \le m$, for which A_i's arise from the solution to the adjoint system (2.6) and $\mathcal{A} = \{T_i : i \le m\}$. Then $\mathbf{\Lambda} + \mathcal{A}$ is a substitution tiling and we can define a tile-substitution Ω satisfying

$$\Omega(\mathbf{\Lambda} + \mathcal{A}) = \mathbf{\Lambda} + \mathcal{A}$$

from (2.6). We call $\mathbf{\Lambda} + \mathcal{A}$ the *associated substitution tiling* of $\mathbf{\Lambda}$. Let $\Phi = (\Phi_{ij})$ be a MFS for Ω. For any subset $\mathbf{\Gamma} = (\Gamma_j)_{j \le m} \subset \mathbf{\Lambda}$, $\Phi_{ij}(\Gamma_j) = Q\Gamma_j + \mathcal{D}_{ij}$, for $j \le m$. Let $\Phi(\mathbf{\Gamma}) = (\cup_{j \le m} \Phi_{ij}(\Gamma_j))_{i \le m}$. Then $\Phi_{ij}(\Lambda_j) = Q\Lambda_j + \mathcal{D}_{ij}$, where $i \le m$. For any $k \in \mathbb{Z}_+$ and $x \in \Lambda_j$ with $j \le m$, we let $\Phi^k(x) = \Phi^{k-1}((\Phi_{ij}(x))_{i \le m})$. Note that for any $k \in \mathbb{Z}_+$, $\Phi^k(\Lambda_j) = (Q^k \Lambda_j + (\mathcal{D}^k)_{ij})_{i \le m}$ where

$$(\mathcal{D}^k)_{ij} = \bigcup_{n_1, n_2, \dots, n_{(k-1)} \le m} (\mathcal{D}_{in_1} + Q\mathcal{D}_{n_1 n_2} + \cdots + Q^{k-1} \mathcal{D}_{n_{(k-1)} j})$$

and $\Phi^k(\mathbf{\Lambda}) = \mathbf{\Lambda}$.

In [22, Lemma 3.2] it is shown that if $\mathbf{\Lambda}$ is a substitution Delone multi-colour set, then there is a finite multi-colour set (cluster) $\mathbf{P} \subset \mathbf{\Lambda}$ for which $\Phi^{n-1}(\mathbf{P}) \subset \Phi^n(\mathbf{P})$ for $n \ge 1$ and $\mathbf{\Lambda} = \lim_{n \to \infty} \Phi^n(\mathbf{P})$. We call such a multi-colour set $\mathbf{P}$ a *generating set* for $\mathbf{\Lambda}$.

Theorem 2.3. [26] *Let $\mathbf{\Lambda}$ be a repetitive primitive substitution Delone multi-colour set in $\mathbb{R}^d$. Then every $\mathbf{\Lambda}$-cluster is legal if and only if $\mathbf{\Lambda}$ is representable.*

On the other hand, if a self-affine tiling $\mathcal{T} = \{T_j + \Lambda_j : j \le m\}$ is given, we get an associated substitution Delone multi-colour set $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \le m}$ of $\mathcal{T}$ (see [23, Lemma 5.4]).

2.4. **Pure point spectrum and overlap coincidence.** Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$. We define the space of tilings as the orbit closure of $\mathcal{T}$ under the translation action: $X_{\mathcal{T}} = \overline{\{-h + \mathcal{T} : h \in \mathbb{R}^d\}}$, in the well-known "local topology": for a small $\epsilon > 0$ two point sets $\mathcal{S}_1, \mathcal{S}_2$ are ϵ-close if $\mathcal{S}_1$ and $\mathcal{S}_2$ agree on the ball of radius ϵ^{-1} around the origin, after a translation of size less than ϵ. The group $\mathbb{R}^d$ acts on $X_{\mathcal{T}}$ by translations which are obviously homeomorphisms, and we get a topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. Let μ be an ergodic invariant Borel probability measure for the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. We consider the associated group of unitary operators $\{U_g\}_{g \in \mathbb{R}^d}$ on $L^2(X_{\mathcal{T}}, \mu)$:

$$U_g f(\mathcal{S}) = f(-g + \mathcal{S}).$$

A vector $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ is said to be an eigenvalue for the $\mathbb{R}^d$-action if there exists an eigenfunction $f \in L^2(X_{\mathcal{T}}, \mu)$, that is, $f \not\equiv 0$ and

$$U_g f = e^{2\pi i g \cdot \alpha} f, \quad \text{for all} \ \ g \in \mathbb{R}^d.$$

The dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ is said to have *pure point(or pure discrete) spectrum* if the linear span of the eigenfunctions is dense in $L^2(X_{\mathcal{T}}, \mu)$. Recall that a topological dynamical system of a self-affine tiling is *uniquely ergodic* i.e. there is a unique invariant probability measure [26].

2.5. **Overlaps.** Overlap and overlap coincidence are originally defined with tiles in substitution tilings [41]. For computational reason, we define overlaps with the corresponding representative points of tiles here. A triple (u, y, v), with $u + T_i, v + T_j \in \mathcal{T}$ and $y \in \Xi(\mathcal{T})$, is called an *overlap* (or *real overlap*) if

$$(u + A_i - y)^\circ \cap (v + A_j)^\circ \neq \emptyset,$$

where $A_i = \mathrm{supp}(T_i)$ and $A_j = \mathrm{supp}(T_j)$. We define $(u + A_i - y) \cap (v + A_j)$ the *support of an overlap* (u, y, v) and denote it by $\mathrm{supp}(u, y, v)$. We say that two overlaps (u, y, v) and (u', y', v') are *equivalent* if there exists $g \in \mathbb{R}^d$ such that $u - y = g + u' - y'$ and $v = g + v'$, where $u + T_i, u' + T_i \in \mathcal{T}$ and $v + T_j, v' + T_j \in \mathcal{T}$ for some $1 \le i, j \le m$. Denote by $[(u, y, v)]$ the equivalence class of an overlap. An overlap (u, y, v) is a *coincidence* if

$$u - y = v \text{ and } u + T_i, v + T_i \in \mathcal{T} \text{ for some } i \le m.$$

Let $\mathcal{O} = (u, y, v)$ be an overlap in $\mathcal{T}$, we define *k-th inflated overlap*

$$\Phi^k \mathcal{O} = \{(u', Q^k y, v') : u' \in \Phi^k(u), v' \in \Phi^k(v), \text{ and } (u', Q^k y, v') \text{ is an overlap}\}.$$

Definition 2.4. We say that a self-affine tiling $\mathcal{T}$ admits an *overlap coincidence* if there exists $\ell \in \mathbb{Z}_+$ such that for each overlap $\mathcal{O}$ in $\mathcal{T}$, $\Phi^\ell \mathcal{O}$ contains a coincidence.

Theorem 2.5. [26, 23] *Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ such that $\Xi(\mathcal{T})$ is a Meyer set. Then $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has a pure point dynamical spectrum if and only if $\mathcal{T}$ admits an overlap coincidence.*

In actual computation, it is not easy to determine whether a given triple is an overlap, since two points can be very close without having the interiors of the corresponding tiles meet. So we introduce a notion of *potential overlaps.*

Let $\xi \in \mathbb{R}^d$ be a fixed point under the substitution such that $\xi \in \Phi_{ii}(\xi)$ for some $i \le m$. When there is no confusion, we will identify ξ with a coloured point (ξ, i) in $\mathbf{\Lambda}_{\mathcal{T}}$. So we write $\xi \in \Phi(\xi)$. We find a basis of $\mathbb{R}^d$

$$\mathcal{B} = \{\alpha_1, \ldots, \alpha_d\} \subset \Xi(\mathcal{T})$$

such that

$$\xi + \alpha_1, \ldots, \xi + \alpha_d \in \Phi^\ell(\xi) \quad \text{for some } \ell \in \mathbb{Z}_+. \tag{2.7}$$

Let

$$\alpha_{max} := \max\{|\alpha_i| : \alpha_i \in \mathcal{B}\}. \tag{2.8}$$

For any $n \in \mathbb{Z}_+$, let

$$e^{(n)} := \max\{|d_{ij} - d'_{k\ell}| : d_{ij} \in (\mathcal{D}^n)_{ij}, d'_{k\ell} \in (\mathcal{D}^n)_{k\ell}, \text{ where } 1 \le i, j, k, \ell \le m\},$$

Let $\|\cdot\|$ be the operator norm induced by Euclidean norm. Since Q is an expansive map, we can find $k \in \mathbb{Z}_+$ such that

$$\|Q^{-k}\| < 1. \tag{2.9}$$

Note that for any $v \in \mathbb{R}^d$,

$$|Q^k v| \ge \frac{1}{\|Q^{-k}\|} |v|. \tag{2.10}$$

Let

$$R = \frac{e^{(k)} \cdot \|Q^{-k}\|}{1 - \|Q^{-k}\|}. \tag{2.11}$$

We say that a triple (u, y, v), with $u + T_i, v + T_j \in \mathcal{T}$ for some $i, j \le m$ and $y \in \Xi(\mathcal{T})$, is called a *potential overlap* if

$$|u - y - v| \le R \tag{2.12}$$

and we say that the potential overlap (u, y, v) occurs by the translation y.

Lemma 2.6. *If (u, y, v) is an overlap, then (u, y, v) is a potential overlap.*

Proof. From (2.6), we get

$$Q^k A_j = \bigcup_{i=1}^{m} ((\mathcal{D}^k)_{ij} + A_i), \quad j \le m.$$

For any $i \le m$ and $a \in A_i$, we can write

$$a = Q^{-k} d_{i_1 i} + Q^{-2k} d_{i_2 i_1} + \cdots, \quad \text{where } d_{i_{n+1} i_n} \in (\mathcal{D}^k)_{i_{n+1} i_n}.$$

Thus for any $i, j \le m$, $a \in A_i$, and $b \in A_j$,

$$|a - b| \le \sum_{n=1}^{\infty} \|Q^{-k}\|^n |d_{i_n i_{n-1}} - d'_{i_n i_{n-1}}| \le \frac{e^{(k)} \cdot \|Q^{-k}\|}{1 - \|Q^{-k}\|}.$$

If (u, y, v) is an overlap where $u + T_i, v + T_j \in \mathcal{T}$ for some $i, j \le m$, then

$$(u + A_i - y) \cap (v + A_j) \ne \emptyset.$$

Let $z \in (u + A_i - y) \cap (v + A_j)$. Then $z - u + y \in A_i$ and $z - v \in A_j$. So

$$|u - y - v| \le \frac{e^{(k)} \cdot \|Q^{-k}\|}{1 - \|Q^{-k}\|} = R.$$

□

Similarly to the k-th iterated overlap, for each potential overlap $\mathcal{O} = (u, y, v)$ in $\mathcal{T}$, we define *k-th inflated potential overlap*

$$\Phi^k \mathcal{O} = \{(u', Q^k y, v') : u' \in \Phi^k(u), v' \in \Phi^k(v), \text{ and } (u', Q^k y, v') \text{ is a potential overlap}\}$$

and the equivalence class of $\Phi^k \mathcal{O}$

$$[\Phi^k \mathcal{O}] = \{[\mathcal{O}'] : \mathcal{O}' \in \Phi^k \mathcal{O}\}.$$

For the computation of overlap coincidence, it is important to have the Meyer property of $\Xi(\mathcal{T})$. The next theorem gives a criterion on Q for the Meyer property. A set of algebraic integers $\Theta = \{\theta_1, \cdots, \theta_r\}$ is a *Pisot family* if for any $1 \le j \le r$, every Galois conjugate γ of θ_j with $|\gamma| \ge 1$ is contained in Θ.

Theorem 2.7. [28] *Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ with a diagonalizable expansion map Q. Suppose that all the eigenvalues of Q are algebraic conjugates with the same multiplicity. Then $\Xi(\mathcal{T})$ is a Meyer set if and only if the set of all the eigenvalues of Q is a Pisot family.*

Summarizing the results of this paper, we provide an algorithm to determine the pure point spectrum of a substitution tiling dynamical system. Let G be a subset of a set of potential overlaps. We construct a graph with multiplicities for G viewing potential overlaps as vertices and defining multiple edges by counting the vertices in the inflated potential overlaps and give the same name G for the graph.

Algorithm : We assume that $\mathcal{T}$ is a self-affine tiling in $\mathbb{R}^d$ with expansion map Q for which $\Xi(\mathcal{T})$ is a Meyer set and $T_i = (A_i, i)$, $i \le m$, are prototiles such that

$$QA_j = \bigcup_{i \le m} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le m.$$

- **Input:** Φ is an $m \times m$ matrix whose each entry is a set of functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $\Phi = (\Phi_{ij})$, where $\Phi_{ij} = \{f : x \to Qx + d, d \in \mathcal{D}_{ij}\}$, i.e. Φ is a MFS for $\mathcal{T}$.
- **Output:** True, if and only if $\mathcal{T}$ has pure point spectrum.

(1) Find an initial point x such that $x \in \Phi_{ii}(x)$ for some $i \le m$.
(2) Find a basis $\{\alpha_1, \dots, \alpha_d\} \subset \mathbb{R}^d$ such that $\alpha_k \in \bigcup_{i \le m}((\Phi^n(x))_i - (\Phi^n(x))_i)$ for some $n \in \mathbb{Z}_+$, for each $1 \le k \le d$.
(3) For each $1 \le k \le d$, find all the potential overlaps $\mathcal{G}_{\alpha,0}$ which occur from the translation α_k.
(4) Find all the potential overlaps $\mathcal{G}$ which occur from the translations $Q^n\alpha_k$ with $1 \le k \le d$ and $n \in \mathbb{Z}_+$.
(5) Find all the potential overlaps $\mathcal{G}_{coin}$ which lead to coincidences within $\sharp\mathcal{G}$-iterations.
(6) If $\rho(\mathcal{G}_{coin}) > \rho(\mathcal{G} \setminus \mathcal{G}_{coin})$, where $\rho(G)$ is the spectral radius of the graph G, return true. Else, return false.

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3. Computing coincidence

In the rest of the paper, we assume that $\mathcal{T}$ is a self-affine tiling in $\mathbb{R}^d$ such that $\Xi(\mathcal{T})$ is a Meyer set. We can choose a representing point of each tile to be in the interior of the tile. In fact, from (2.2), for any $a_i \in A_i$ with $i \le m$, we can get

$$Q(A_j - a_j) = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} - Qa_j + a_i + (A_i - a_i)) \quad \text{for } j \le m. \tag{3.1}$$

We may consider new prototiles

$$\{T_1 - a_1, \dots, T_m - a_m\}$$

with new digit sets

$$\mathcal{D}'_{ij} = \mathcal{D}_{ij} - Qa_j + a_i. \tag{3.2}$$

Without loss of generality we can assume that for any $T_i = (A_i, i) \in \mathcal{A}$, $i \le m$,

$$0 \in A_i.$$

Note that the choice of the representing point does not change the translation distance between two tiles of the same type.

3.1. **Meyer sets.** Let Λ be a Meyer set and $[\Lambda]$ be the Abelian group generated by Λ. Then $[\Lambda]$ is finitely generated. So $[\Lambda] = \oplus_{i=1}^{s} \mathbb{Z} e_i$. We define $||\cdot|| : [\Lambda] \to \mathbb{N}$ such that $||\sum_{i=1}^{s} n_i e_i|| = \sum_{i=1}^{s} |n_i|$. For each positive integer n, let

$$F(n) := \{u \in [\Lambda] \, : \, ||u|| \le n\}.$$

Note that $F(n)$ is finite. Choose $h > 0$ such that every open ball of radius h in $\mathbb{R}^d$ meets at least one element in Λ. Since $\Lambda - \Lambda := \{x - y : x, y \in \Lambda\}$ is uniformly discrete from the Meyer property of Λ, we let $L \in \mathbb{Z}_+$ be an upper bound for the number of points in $\Lambda - \Lambda$ that can lie in an open ball of radius $2h$. Let

$$\ell := \max\{||u|| \, : \, u \in \Lambda - \Lambda, |u| < 3h\}.$$

Proposition 3.1. [19, 33] *Let* Λ *be a Meyer set. Then*

$$\Lambda - \Lambda \subset \Lambda + F, \quad \textit{where } F = F(2\ell(L-1)).$$

It is proved in [26, Lemma A.8] that the number of equivalence classes of overlaps for a tiling which has the Meyer property is finite. We apply the same argument to get the number of equivalence classes of potential overlaps for a tiling and give an explicit upper bound for the number.

Lemma 3.2. *Let* $\mathcal{T}$ *be a self-affine tiling and* $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i\le m}$ *be the associated substitution Delone multi-colour set of* $\mathcal{T}$*. Let* $\Lambda = \bigcup_{i\le m} \Lambda_i$*. Suppose that* Λ *is a Meyer set. The number of equivalence classes of potential overlaps for* $\mathcal{T}$ *is less than or equal to* $m^2 I$*, where*

$$I = \#((\Lambda + F + F + F) \cap B_R(0)),$$

with $F = F(2\ell(L-1))$ *as in Prop. 3.1.*

Proof. Let (u, y, v) be a potential overlap in $\mathcal{T}$ for which $u+T_i, v+T_j \in \mathcal{T}$. Then $|u-y-v| \le R$. Note that $u - y - v \in (\Lambda - \Lambda) - (\Lambda - \Lambda)$. From Prop. 3.1,

$$\begin{aligned}(\Lambda - \Lambda) - (\Lambda - \Lambda) &\subset \Lambda + F - (\Lambda + F)\\ &\subset \Lambda + F + F + F\,.\end{aligned}$$

The equivalence classes of the potential overlaps are completely determined by i, j and the vector $u - y - v$. Thus the claim follows. □

Remark 1. Let $\mathbf{\Lambda} = (\Lambda_i)_{i\le m}$ be a substitution Delone multi-colour set for which $\Lambda = \cup_{i\le m}\Lambda_i$ is a lattice. It has been shown in [39] that modular coincidence, which is equivalent to the overlap coincidence in lattice substitution Delone multi-colour sets, can be determined within an exponential bound $2^m - m - 2$. Note that there are only $m^2 I$ number of potential overlaps where $I = \#(\Lambda \cap B_R(0))$, since $\Lambda - \Lambda = \Lambda$. In fact, there are at most $\frac{m(m+1)}{2}$ number of overlaps in $\mathcal{T}_{\mathbf{\Lambda}}$, since all tiles have congruent supports and so all overlaps are formed by order insensitive pairs of tiles. Overlap coincidence for $\mathcal{T}_{\mathbf{\Lambda}}$ can be determined within $\frac{m(m-1)}{2}$ number of iterations of each overlap. However to observe the modular coincidence, we need to iterate more. For 1-dimension lattice substitution Delone sets[1], a polynomial bound $\frac{m^3-m}{6}$ seems to be the lowest bound known so far for the occurrence of modular coincidence. In fact, this problem is equivalent to finding the best upper bound for the length of synchronizing words in deterministic m states automata. Černý's conjecture says that the best bound would be $(m-1)^2$ (see [45]).

2. We note from [27, Th. 4.14] that Λ is a Meyer set if and only if $\Xi(\mathcal{T})$ is a Meyer set in the self-affine tiling $\mathcal{T}$.

3.2. **Coincidence and computation.** From now on, we assume that $\mathcal{T}$ is a self-affine tiling with an expansion map Q for which $\Xi(\mathcal{T})$ is a Meyer set.

For $\alpha \in \Xi(\mathcal{T})$, define

$$\mathcal{E}_\alpha := \{(u, Q^n\alpha, v) \,:\, (u, Q^n\alpha, v) \text{ is overlap in } \mathcal{T},\ n \in \mathbb{N}\ \}.$$

For any $n \in \mathbb{Z}_{\ge 0}$, define

$$D_{Q^n\alpha} := \mathcal{T} \cap (\mathcal{T} - Q^n\alpha)$$

and

$$\text{dens}(D_{Q^n\alpha}) = \lim_{n\to\infty} \frac{\text{Vol}(D_{Q^n\alpha} \cap B_n)}{\text{Vol}(B_n)}\,.$$

[1] We thank Dirk Frettlöh, Johan Nilsson, and Wolfgang Steiner for the following comment.

The following lemma is proved in [26, Lemma A.9] with the subdivision graph for overlaps. The third statement in [26, Lemma A.9] is stated for each overlap having coincidence in some iteration. However, since there are only finite number of overlaps for the tiling, we can restate the statement for any overlap as follows.

Lemma 3.3. [26, Lemma A.9] *Let $\alpha \in \Xi(\mathcal{T})$. The following are equivalent:*

(i) $\lim_{n\to\infty} \mathrm{dens}(D_{Q^n\alpha}) = 1$.
(ii) $1 - \mathrm{dens}(D_{Q^n\alpha}) \le br^n$ *for any $n \ge 1$, for some $b > 0$ and $r \in (0,1)$.*
(iii) *There exists $\ell \in \mathbb{Z}_+$ such that for any overlap $\mathcal{O}$ in $\mathcal{E}_\alpha$, $\Phi^\ell\mathcal{O}$ contains a coincidence.*

The next theorem is basically in [41] and [26, Th. 4.7]. We notice here that we only need to consider the overlaps in $\mathcal{E}_\alpha$ for all $\alpha \in \mathcal{B}$ to check the overlap coincidence of $\mathcal{T}$. We rewrite the theorem in the form that we use here.

Theorem 3.4. [41], [26, Th. 4.7] *Let $\mathcal{T}$ be a self-affine tiling for which $\Xi(\mathcal{T})$ is a Meyer set. Then there exists $\ell \in \mathbb{Z}_+$ such that for any $\alpha \in \mathcal{B}$ and any overlap $\mathcal{O} \in \mathcal{E}_\alpha$, $\Phi^\ell\mathcal{O}$ contains a coincidence if and only if $\mathcal{T}$ admits an overlap coincidence.*

Proof. We only prove the sufficiency direction, since the other direction is clear. Suppose that for any $\alpha \in \mathcal{B}$ and any overlap $\mathcal{O} \in \mathcal{E}_\alpha$, $\Phi^\ell\mathcal{O}$ contains a coincidence. From the argument of [26, Lemma A.9], for some $b > 0$ and $r \in (0,1)$

$$1 - \mathrm{dens}(D_{Q^n\alpha}) \le br^n \quad \text{for any } n \in \mathbb{N}\,.$$

Hence

$$\sum_{n=0}^{\infty}(1 - \mathrm{dens}(D_{Q^n\alpha})) < \infty\,.$$

Since $\mathcal{B}$ forms a basis for $\mathbb{R}^d$, by [41, Thm. 6.1] the dynamical system of Delone multi-colour set has pure point spectrum. By [26, Thm. 4.7 and Lemma A.9], $\mathcal{T}$ admits an overlap coincidence. □

In order to find first all equivalent classes of potential overlaps which occur from the translations of α_i for any $1 \le i \le d$, we want to know how much region of the intersection of $\mathcal{T}$ and $\mathcal{T} - \alpha_i$, $1 \le i \le d$, we have to look. We use same notations for points with colours in $\mathbf{\Lambda}_\mathcal{T}$ and points in $\mathbb{R}^d$. This should not cause any confusion.

Let

$$\mathcal{J}(\mathbf{\Lambda}_\mathcal{T}) = \{[\mathbf{P}] : \mathbf{P} = \{y, z\} \subset \mathbf{\Lambda}_\mathcal{T} \;\text{ satisfies }\; |y - z| < R + \|Q^{-k}\|\alpha_{max}\}\,,$$

where k and α_{max} are defined as (2.8) and (2.9), respectively. Let

$$\mathcal{J}(\mathbf{\Gamma}) := \{[\mathbf{P}] \in \Theta : \mathbf{P} \subset \mathbf{\Gamma}\}, \quad \text{where } \mathbf{\Gamma} \subset \mathbf{\Lambda}_\mathcal{T}\,.$$

Lemma 3.5. *If $\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\Phi^{N+k}(\xi))$ for some $N \in \mathbb{Z}_+$, then*

$$\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\mathbf{\Lambda}_\mathcal{T})\,.$$

Proof. Let $\mathbf{P}$ be a cluster in $\Phi^{N+k+1}(\xi)$ such that $[\mathbf{P}] \in \mathcal{J}(\mathbf{\Lambda}_\mathcal{T})$. There must be a cluster $\mathbf{P}' = \{y, z\} \subset \Phi^{N+1}(\xi)$ satisfying $\mathbf{P} \subset \Phi^k(\mathbf{P}')$. We claim that $[\mathbf{P}'] \in \mathcal{J}(\mathbf{\Lambda}_\mathcal{T})$. We only need to show that $|y - z| \le R + \|Q^{-k}\|\alpha_{max}$. Suppose that $|y - z| > R + \|Q^{-k}\|\alpha_{max}$. Then

$$|Q^k y - Q^k z| \ge \frac{1}{\|Q^{-k}\|}|y - z| > \frac{1}{\|Q^{-k}\|}(R + \|Q^{-k}\|\alpha_{max}) = \frac{R}{\|Q^{-k}\|} + \alpha_{max}.$$

For any $y' \in \Phi^k(y)$ and $z' \in \Phi^k(z)$, $y' = Q^k y + d_1$ and $z' = Q^k z + d_2$ for some $d_1, d_2 \in \cup_{i,j\le m}(\mathcal{D}^k)_{ij}$. Thus

$$\begin{aligned} |y'-z'| &= |Q^k y - Q^k z + d_1 - d_2| \ge |Q^k y - Q^k z| - e^{(k)} \\ &> \frac{R}{\|Q^{-k}\|} + \alpha_{max} - e^{(k)} = R + \alpha_{max} \quad \text{from (2.11)} \\ &> R + \|Q^{-k}\|\alpha_{max}. \end{aligned}$$

It contradicts to the choice of $\mathbf{P}$. Hence $[\mathbf{P}'] \in \mathcal{J}(\mathbf{\Lambda}_{\mathcal{T}})$. From the assumption, note that

$$\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\Phi^{N+1}(\xi)) = \cdots = \mathcal{J}(\Phi^{N+k}(\xi)).$$

So there exists a cluster $\mathbf{P}''$ in $\Phi^N(\xi)$ which is equivalent to $\mathbf{P}'$. Then $\Phi^k(\mathbf{P}'')$ contains a cluster which is equivalent to the cluster $\mathbf{P}$. Thus

$$\mathcal{J}(\Phi^{N+k}(\xi)) = \mathcal{J}(\Phi^{N+k+1}(\xi)).$$

Hence

$$\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\lim_{n\to\infty} \Phi^n(\xi)).$$

By the repetitivity of $\mathbf{\Lambda}_{\mathcal{T}}$, all the clusters in $\mathbf{\Lambda}_{\mathcal{T}}$ whose equivalent classes are in $\mathcal{J}(\mathbf{\Lambda}_{\mathcal{T}})$ should occur in $\mathcal{J}(\lim_{n\to\infty}\Phi^n(\xi))$. Therefore $\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\mathbf{\Lambda}_{\mathcal{T}})$. □

Lemma 3.6. *If $\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\mathbf{\Lambda}_{\mathcal{T}})$, then for each $\alpha \in \mathcal{B}$,*

$$\mathcal{G}_{\alpha,0} := \{[(y,\alpha,z)] : y, z \in \Phi^{N+k}(\xi) \text{ and } |y-\alpha-z| < R\}$$

contains all the different equivalence classes of potential overlaps which can occur from the translation of α.

Proof. If $\{y,z\} \subset \mathbf{\Lambda}_{\mathcal{T}}$ such that $|y-\alpha-z| < R$ for $\alpha \in \mathcal{B}$, there exist $u, v \in \mathbf{\Lambda}_{\mathcal{T}}$ such that $y \in \Phi^k(u)$ and $z \in \Phi^k(v)$. Let $y = Q^k u + d_1$ and $z = Q^k v + d_2$, where $d_1, d_2 \in \cup_{i,j\le m}(\mathcal{D}^k)_{ij}$. Let $\mathbf{P} = \{u, v\}$. We claim that $[\mathbf{P}] \in \mathcal{J}(\Phi^N(\xi))$. Suppose that $|u-v| > R + \|Q^{-k}\|\alpha_{max}$. Then

$$\begin{aligned} |y-z| &= |Q^k u - Q^k v + d_1 - d_2| \ge |Q^k(u-v)| - e^{(k)} \\ &\ge \frac{1}{\|Q^{-k}\|}|u-v| - e^{(k)} > \frac{R}{\|Q^{-k}\|} + \alpha_{max} - e^{(k)} = R + \alpha_{max} \end{aligned}$$

It contradicts to the choice of $\{y,z\} \subset \mathbf{\Lambda}_{\mathcal{T}}$. Since $\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\mathbf{\Lambda}_{\mathcal{T}})$, $[(y,\alpha,z)] \in \mathcal{G}_{\alpha,0}$. □

Lemma 3.7. *If $\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\mathbf{\Lambda}_{\mathcal{T}})$ for some $N \in \mathbb{Z}_+$, then for each $n \in \mathbb{Z}_{\ge 0}$ and $\alpha \in \mathcal{B}$,*

$$\{[(y, Q^n\alpha, z)] : y, z \in \Phi^{N+k+n}(\xi) \text{ and } (y, Q^n\alpha, z) \text{ is a potential overlap}\}$$

contains all the different equivalence classes of potential overlaps which occur from the translation of $Q^n\alpha$.

Proof. We argue this by induction. Note that when $n = 0$, the claim is true. Suppose that it is true for $n = i$, $i \in \mathbb{Z}_+$. Consider $n = i+1$. Let $\mathcal{O}$ be a potential overlap which occurs from the translation of $Q^{i+1}\alpha$. Then there exist $y, z \in \Phi^{N+i+s}(\xi) \subset \mathbf{\Lambda}$ for some $s \in \mathbb{N}$ such that

$$\mathcal{O} = [(y, Q^{i+1}\alpha, z)].$$

Then there exists a potential overlap (u, α, v) with $u, v \in \Phi^{N+i+s-1}(\xi)$ such that

$$(y, Q^{i+1}\alpha, z) \in \Phi(u, Q^i\alpha, v).$$

But we know that there exists a potential overlap $(u', Q^i\alpha, v')$ with $u', v' \in \Phi^{N+k+i}(\xi)$ which is equivalent to a potential overlap $(u, Q^i\alpha, v)$ by the assumption. Thus there exists an equivalent potential overlap $(y', Q^{i+1}\alpha, z')$ to $(y, Q^{i+1}\alpha, z)$ which is contained in $\Phi(u', Q^i\alpha, v')$. Note that

$$y', z' \in \Phi^{N+k+i+1}(\xi)\,.$$

Thus

$$\{[(y, Q^{i+1}\alpha, z)] : y, z \in \Phi^{N+k+i+1}(\xi)\}$$

contains all the different types of equivalent potential overlaps which occur from the translation $Q^{i+1}\alpha$. Thus the claim is proved. □

Suppose that $\mathcal{J}(\Phi^N(\xi)) = \mathcal{J}(\mathbf{\Lambda}_\mathcal{T})$ for some $N \in \mathbb{Z}_+$. For any $\alpha \in \mathcal{B}$ and any $M \in \mathbb{Z}_{\geq 0}$, define

$$\mathcal{G}_{\alpha,M} := \bigcup_{0 \leq n \leq M} \{[(y, Q^n\alpha, z)] : y, z \in \Phi^{N+k+n}(\xi) \text{ and } (y, Q^n\alpha, z) \text{ is a potential overlap}\}$$

$$\mathcal{G}_\alpha := \bigcup_{M \in \mathbb{Z}_{\geq 0}} \mathcal{G}_{\alpha,M} \quad \text{and} \quad \mathcal{G} = \bigcup_{\alpha \in \mathcal{B}} \mathcal{G}_\alpha.$$

Lemma 3.8. *Let $\alpha \in \mathcal{B}$. If $\mathcal{G}_{\alpha,M} = \mathcal{G}_{\alpha,M+1}$ for some $M(= M_\alpha) \in \mathbb{Z}_{\geq 0}$, then*

$$\mathcal{G}_{\alpha,M} = \mathcal{G}_\alpha.$$

Proof. Let $[(y, Q^n\alpha, z)] \in \mathcal{G}_{\alpha,M+2}$, where $0 \leq n \leq M+2$. Then there exists a potential overlap $(y', Q^{n-1}\alpha, z')$, with $y', z' \in \Phi^{N+k+M+1}(\xi)$, such that

$$(y, Q^n\alpha, z) \in \Phi(y', Q^{n-1}\alpha, z').$$

Since $\mathcal{G}_{\alpha,M} = \mathcal{G}_{\alpha,M+1}$, $(y', Q^{n-1}\alpha, z')$ is equivalent to $(y'', Q^{n'}\alpha, z'')$ for some $y'', z'' \in \Phi^{N+k+n'}(\xi)$ where $0 \leq n' \leq M$. So

$$[\Phi(y', Q^{n-1}\alpha, z')] = [\Phi(y'', Q^{n'}\alpha, z'')]\,.$$

Thus $[(y, Q^n\alpha, z)] \in \mathcal{G}_{\alpha,M+1}$. □

Let $\mathcal{H}$ be the set of all equivalent classes of overlaps in $\mathcal{T}$ and $\mathcal{G}_{coin}$ be the set of all equivalence classes of overlaps in $\mathcal{T}$ which lead to coincidence after some iterations. Note that

$$\mathcal{G}_{coin} \subset \mathcal{H} \subset \mathcal{G}$$

and for each $\alpha \in \mathcal{B}$, $M_\alpha \leq m^2 I$, since there are at most $m^2 I$ equivalence classes of potential overlaps in $\mathcal{T}$ by the Lemma 3.2.

Theorem 3.9. *If an overlap in $\mathcal{T}$ has a coincidence after some iterations, it should happen before $\sharp\mathcal{G}$ number of iterations. In other words,*

$$\begin{aligned}\mathcal{G}_{coin} \;=\; & \{[(u, Q^n\alpha, v)] \in \mathcal{G} \,:\, \Phi^t_{\ell i}(u - Q^n\alpha) \cap \Phi^t_{\ell j}(v) \neq \emptyset \\ & \textit{for some } 1 \leq \ell \leq m \textit{ and } 0 \leq t < \sharp\mathcal{G}, \textit{ where } u + T_i, v + T_j \in \mathcal{T}\}.\end{aligned}$$

Proof. Note that there are at most $\sharp\mathcal{G}$ number of equivalence classes of potential overlaps in $\mathcal{T}$. For any overlap $\mathcal{O}$, if coincidence does not occur in $\Phi^t(\mathcal{O})$ for some $0 \leq t \leq \sharp\mathcal{G}$, coincidence will never occur in $\Phi^n(\mathcal{O})$ for any $n \in \mathbb{Z}_{\geq 0}$. Since Q is an expansive map, it is sufficient to check for $0 \leq t < \sharp\mathcal{G}$.

4. The potential overlaps that are not real overlaps

We aim to prove the following Theorem 4.1 in this section. The algorithm given by this theorem is quite simple and easy to implement and applies to all self-affine tilings whenever the expansion maps Q and digit sets $\mathcal{D}_{ij}$, which define the self-affine tilings, are given.

In the sequel, we construct a *graph with multiplicities* viewing potential overlaps in $\mathcal{G}$ as vertices and define multiple edges by counting the vertices in the inflated potential overlaps. Hereafter we deal with the representatives of equivalence classes of potential overlaps in $\mathcal{G}$. Let (u, y, v) be a potential overlap, where $u + T_i, v + T_j \in \mathcal{T}$, $T_i = (A_i, i)$ and $T_j = (A_j, j)$. Inflating the corresponding tiles in the potential overlap (u, y, v) and intersecting them, we observe

$$\begin{aligned} & Q(u + A_i - y) \cap Q(v + A_j) \\ (4.1) \quad = & \bigcup_{k=1}^{m} (A_k + \mathcal{D}_{ki} + Qu - Qy) \cap \bigcup_{\ell=1}^{m} (A_\ell + \mathcal{D}_{\ell j} + Qv) \\ = & \bigcup_{k=1}^{m} \bigcup_{\ell=1}^{m} \bigcup_{d_{ki} \in \mathcal{D}_{ki}} \bigcup_{d_{\ell j} \in \mathcal{D}_{\ell j}} (((A_k + d_{ki} - d_{\ell j} + Qu - Qy - Qv) \cap A_\ell) + d_{\ell j} + Qv). \end{aligned}$$

The equivalence class $[(u, y, v)]$ can be viewed as an element (i, y, j) where $1 \leq i, j \leq m$ and $z = u - y - v$ with $|z| \leq R$ where R is as defined in (2.11). We define the multiple edge

$$(i, z, j) \xrightarrow{e} (k, z', \ell) \tag{4.2}$$

if $z' = d_{ki} - d_{\ell j} + Qx_i - Qy - Qx_j$ with $|z'| \leq R$ for $d_{ki} \in \mathcal{D}_{ki}$ and $d_{\ell j} \in \mathcal{D}_{\ell j}$, where the multiplicities of the edge is given by $\#\{(d_{ki}, d_{\ell j}) \in \mathcal{D}_{ki} \times \mathcal{D}_{\ell j} \mid z' = d_{ki} - d_{j\ell} + Qu - Qy - Qv\}$. Keeping the multiplicity in the graph is essential to distinguish real overlaps from potential overlaps that are not. Recall that (i, y, j) is a coincidence if $i = j$ and $y = 0$. We consider $\mathcal{G}_{coin}$ as the induced graph of $\mathcal{G}$ to the vertices which have a path leading to a coincidence. Also we define $\mathcal{G}_{res}$ by the induced graph generated by the complement of such set from $\mathcal{G}$, i.e. $\mathcal{G}_{res} = \mathcal{G} \setminus \mathcal{G}_{coin}$. For any graph G, we denote by $\rho(G)$ the spectral radius of the graph G.

Theorem 4.1. *Let $\mathcal{T}$ be a self-affine tiling for which $\Xi(\mathcal{T})$ is a Meyer set. Then the following are equivalent;*

(i) *$\mathcal{T}$ admits an overlap coincidence.*
(ii) *$\rho(\mathcal{G}_{coin}) > \rho(\mathcal{G}_{res})$.*

The potential overlaps (u, y, v) can be divided into three cases.

- No intersection overlap: $(u + A_i - y) \cap (v + A_j) = \emptyset$,
- Boundary touching overlap: $u + A_i - y$ and $v + A_j$ are just touching at their boundaries,
- Real overlap: $(u + A_i - y)^\circ \cap (v + A_j)^\circ \neq \emptyset$.

If $(u + A_i - y) \cap (v + A_j)$ is empty, then the distance between two tiles $u + T_i - y$ and $v + T_j$ becomes larger by the iterations of the tile-substitution Ω. Therefore this potential overlap does not produce an infinite walk on the graph of potential overlaps by the iterations of Ω. However, when they are touching at their boundaries, this gives infinite walks on the graph and it may contribute to the number of possible paths and consequently to the spectral radius by repeated inflation. Our task is to prove that this contribution is small so that we can distinguish them from real overlaps.

Let (V, Γ) be a directed graph with a vertex set $V = \{1, \ldots, M\}$ and an edge set Γ. We call $\{f_e : e \in \Gamma\}$, a collection of contractions $f_e : \mathbb{R}^d \to \mathbb{R}^d$, a *graph-directed iterated* *function system (GIFS)*. Let $\Gamma_{k\ell}$ be the set of edges from vertex k to ℓ, then there are unique non-empty compact sets $\{E_k\}_{k=1}^N$ satisfying

$$E_k = \bigcup_{\ell=1}^{M} \bigcup_{e\in\Gamma_{k\ell}} f_e(E_\ell), \quad \text{for } k \le M \tag{4.3}$$

(see [30]). We say that (4.3) satisfies the *open set condition* (OSC) if there are open sets U_k so that

$$\bigcup_{\ell=1}^{M} \bigcup_{e\in\Gamma_{k\ell}} f_e(U_\ell) \subset U_k, \quad \text{for } k \le M$$

and the left side is a disjoint union. Further if $U_k \cap E_k \ne \emptyset$ for all $k \le M$, then we say that the GIFS satisfies the *strong open set condition* (SOSC).

We observe from (4.1)

$$\begin{aligned}(A_i + u - y - v) \cap A_j = \\ \bigcup_{k\le m} \bigcup_{\ell\le m} \bigcup_{d_{ki}\in\mathcal{D}_{ki}} \bigcup_{d_{\ell j}\in\mathcal{D}_{\ell j}} Q^{-1}\left(((A_k + d_{ki} - d_{\ell j} + Qu - Qy - Qv) \cap A_\ell) + d_{\ell j}\right).\end{aligned} \tag{4.4}$$

If (k, z', ℓ), where $z' = d_{ki} - d_{\ell j} + Qu - Qy - Qv$, is not a potential overlap, we discard it from (4.4).

Let M be the number of elements in $\mathcal{G}_{res}$. Now we construct a graph for $\mathcal{G}_{res}$ identifying the potential overlaps in $\mathcal{G}_{res}$ with the numbers in $\{1, \cdots, M\}$. In the graph $\mathcal{G}_{res}$, if vertices have no outgoing edges, we can remove them from $\mathcal{G}_{res}$ successively. For each potential overlap (i, v, j) which corresponds to a vertex $k \le M$, let

$$E_k := (A_i + z) \cap A_j, \quad \text{where } z = u - y - v. \tag{4.5}$$

Let $\Gamma_{k\ell}$ be the set of edges from k to ℓ where $1 \le k, \ell \le M$. From (4.4), we notice that E_k's satisfy GIFS (4.3), where $f_e(x) = Q^{-1}(x + d_e)$ and $d_e \in \mathcal{D}_{\ell j}$ for some $\ell \le m$. Denote by $\Gamma^n_{k\ell}$ the set of paths of length n from k to ℓ and for $I = e_1 \dots e_n \in \Gamma^n_{k\ell}$ we put $d_I = \sum_{i=1}^n Q^{n-i} d_{e_i}$.

Remark 4.2. *If $\mathcal{G}_{res}$ does not contain the vertices of real overlaps, each E_k with $k \le M$ is a subset of ∂A_j for some tile $T_j = (A_j, j) \in \mathcal{A}$.*

We use the recent development by He and Lau [15] which slightly modifies the Hausdorff measure. They introduced a new type of gauge function, called *pseudo norm* $w : \mathbb{R}^d \to \mathbb{R}_+$ corresponding to Q having key properties:

$$w(Qx) = |\det(Q)|^{1/d} w(x) \tag{4.6}$$

and

$$w(x + y) \le c(w) \max(w(x), w(y)) \tag{4.7}$$

for some positive constant $c(w)$. This w induces the same topology as Euclidean norm. By w, they modified the definition of Hausdorff measure by: for an open set $U \subset \mathbb{R}^d$, a subset $X \subset \mathbb{R}^d$ and $s, \delta \in \mathbb{R}_+$,

$$\mathrm{diam}_w(U) = \sup_{x,y\in U} w(x - y),$$

$$\mathcal{H}^{s,\delta}_w(X) = \inf_{X\subset\cup_i U_i} \left\{ \sum_i \mathrm{diam}_w(U_i)^s \;\middle|\; \mathrm{diam}_w(U_i) < \delta \right\}$$

and

$$\mathcal{H}^s_w(X) = \lim_{\delta\downarrow 0} \mathcal{H}^{s,\delta}_w(X).$$

Our new Hausdorff dimension is defined by

$$\dim_H^w(X) = \sup\{s|\mathcal{H}_w^s(X) = \infty\} = \inf\{s|\mathcal{H}_w^s(X) = 0\}.$$

Using the pseudo norm w, one can treat self-affine attractors almost as easy as self-similar ones.

To prove Theorem 4.1, we need the next lemma of Luo-Yang [29]. This generalizes a result in [15] and its proof basically follows from the idea of Schief [36], but using pseudo norm instead of Euclidean norm. Note that strong connectedness of GIFS is essential.

Lemma 4.3. [29, Th. 1.1] *Assume that the GIFS is strongly connected. Then the following conditions are equivalent:*

(1) $\{d_I \mid I \in \Gamma_{k\ell}^n\}$ *give distinct* $\#(\Gamma_{k\ell}^n)$ *points whose distance between two points has a uniform lower bound* $r > 0$ *for all* $k, \ell \le M$ *and* $n \ge 1$.
(2) *The GIFS satisfies strong open set condition (SOSC).*

Proof of Theorem 4.1 (ii) $\Rightarrow$ (i). Consider an overlap (u, y, v) where $u + T_i, v + T_j \in \mathcal{T}$. Applying Q^n to the overlapping part of the overlap, we have

$$\mu_d(Q^n((u + A_i - y) \cap (v + A_j))) = |\det Q|^n \mu_d((u + A_i - y) \cap (v + A_j))$$

where μ_d is the d-dim Lebesgue measure. We know $|\det Q| = \beta$ where β is the Perron Frobenius root of substitution matrix $(\#(\mathcal{D}_{ij}))$ (see [22]). Since there are only finitely many overlaps up to translations, there exist $r > 0$ and $R > 0$ such that $(u + A_i - y)^\circ \cap (v + A_j)^\circ$ contains a ball of radius r and is surrounded by a ball of radius R. After n-iteration of inflation, the number of potential overlaps K_n generated from (u, y, v) is estimated:

$$c_1\beta^n \le K_n \le c_2\beta^n$$

with some positive constants c_1 and c_2. Each real overlap gives this growth of potential overlaps. It implies that $\mathcal{G}_{res}$ cannot contain any real overlap (Recall that we are taking into account the multiplicities of overlap growth). This shows the claim.

(i) $\Rightarrow$ (ii). We show that if all overlaps lead to a coincidence then $\mathcal{G}_{res}$ cannot have a spectral radius $\beta = |\det Q| (= \rho(\mathcal{G}_{coin}))$. By the assumption, $\mathcal{G}_{res}$ does not contain overlaps. So from Remark 4.2, $Y = \bigcup_{k=1}^M E_k$, where E_k's are defined as in (4.5), is the subset of the union of boundaries of tiles. By Lemma 4.3 the GIFS satisfies OSC because the uniform discreteness condition (1) of the Lemma 4.3 automatically follows from the fact that $(\mathcal{D}^n)_{ij}$'s are uniformly discrete for any $i, j \in m$ and $n \ge 1$(see [22]).

We follow Mauldin-Williams [30] to compute a new Hausdorff dimension using pseudo norm w instead of Euclidean norm. Let

$$s = d \log \gamma / \log \beta,$$

where $\gamma = \rho(\mathcal{G}_{res})$ and $\beta = |\det Q|$. We study the value $\mathcal{H}_w^s(Y)$. First, assuming strong connectedness of GIFS and OSC, we show $0 < \mathcal{H}_w^s(Y) < \infty$ by using standard mass distribution principle (c.f. Theorem 1.2 in [29]). Second we use a simple fact: an infinite path on GIFS must eventually fall into a single strongly connected component. Thus for GIFS without strong connectedness, we classify infinite walks on GIFS by the prefixes before they fall into the last strongly connected components. This gives an expression of an attractor of general GIFS as a countable union of contracted images of attractors which belong to strongly connected components. In this way we can show the Hausdorff measure $\mathcal{H}_w^s$ is positive and σ-finite, by applying Lemma 4.3 to each strongly connected component. This shows the new Hausdorff dimension of Y with respect to the pseudo norm w

$$\dim_H^w(Y) = s.$$

Notice that the value $\mathcal{H}^s_w(Y)$ can be infinite since we do not know that our GIFS is strongly connected (c.f. [30, Th. 4]).

Now if $s = d$, then $\mathcal{H}^s_w$ is a translationally invariant Borel regular measure having positive value for any open sets because the pseudo norm is comparable with Euclidean norm ([15, Prop. 2.4]). Therefore $\mathcal{H}^s_w$ must be a constant multiple of the d-dimensional Lebesgue measure, by the uniqueness of Haar measure. But this is impossible because the d-dimensional Lebesgue measure of the boundary of self-affine tiles must be 0(see [32]). This shows $s = d \log \gamma / \log \beta < d$ which completes the proof. □

The following conjecture is a folklore. It is mentioned as an open problem in [43] from personal communication with M. Urbański.

Conjecture 4.4. *For d-dimensional non-polygonal self-affine tiling $\mathcal{T}$, each tile $T = (A, i)$ satisfies*

$$d - 1 < \dim_H(\partial A) < d.$$

We partially solve a version of this conjecture in the following Theorem 4.5. Indeed if the matrix Q gives similitudes, this settles the right inequality of Conjecture 4.4.

Theorem 4.5. *For d-dimensional self-affine tiling $\mathcal{T}$, each tile $T = (A, i)$ satisfies*

$$\dim^w_H(\partial A) < d.$$

Proof. We consider a collection of all pairs of tiles in $\mathcal{T}$ whose boundaries are touching. As in (4.2) and (4.4), we get a new GIFS which is defined on this collection. Applying the same argument as in Theorem 4.1, we get $s = d \log \gamma / \log \beta < d$ which shows the claim. □

5. Examples

We implemented Mathematica programs which perform our algorithm to check the overlap coincidence for self-affine tilings. Readers can get the Mathematica programs in the following website.

`http://mathweb.sc.niigata-u.ac.jp/~akiyama/Research1.html`

For a given expanding matrix Q and digit sets $\mathcal{D}_{ij}$ of a self-affine tiling which has the Meyer property, the program gives outputs $\rho(G_{coin})$ and $\rho(G_{res})$. By Theorem 4.1, we can determine whether it satisfies overlap coincidence or not. If the tiling does not satisfy the Meyer property, it may not stop, or stop but produce incorrect outputs.

In actual computation, it is the bottleneck of the program to find all initial potential overlaps for Lemma 3.5 and 3.6. So we use two major tricks in the program to make the computation fast. First, we translate the digits $\mathcal{D}_{ij}$ to $\mathcal{D}'_{ij}$ as shown in (3.2) such that the number of potential overlaps and $e^{(k)}$ are small. The size of $e^{(k)}$ is significant in the computation of collecting all the initial potential overlaps in Lemma 3.5. To make $e^{(k)}$ small, we obtain some number of points in A_i using the tile equation (2.2) and choose a_i among them which is located closest to their centroid. Then we shift tiles A_i to $A_i - a_i$. Second, in order to get all the initial potential overlaps $\mathcal{G}_{\alpha,0}$ in Lemma 3.6, we try to find a fine lattice in $\mathbb{R}^d$ such that we make an embedding of an iterated point set into the lattice taking the closest lattice point for each point of the set. Using the lattice, we can easily compute the candidates of initial potential overlaps, which is much faster than dealing with the original point set. We list selected examples of our computation below.

For an 1-dimension substitution sequence, we can obtain a self-similar tiling by suspension associating to each letter the interval whose length is each entry of a left eigenvector of the incidence matrix of the substitution. Pure point spectrum for the $\mathbb{Z}$-action on a substitution sequence dynamical system is equivalent to pure point spectrum for the $\mathbb{R}$-action

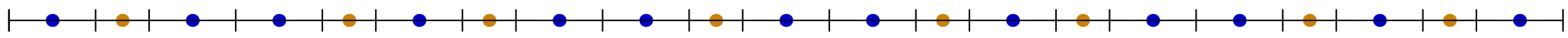


FIGURE 1. Fibonacci substitution tiling

on its suspension tiling dynamical system [8]. The following Example 5.1 and 5.2 show how to obtain a tile substitution when a symbolic letter substitution is given. Since the tile equation is generated automatically in symbolic letter substitutions, we give also a separate computational program in the above website to check overlap coincidence directly from the given symbolic substitution.

We show the results of the following examples in Table 1.

Example 5.1. Fibonacci substitution tiling is a well-known 1-dimension tiling whose substitution is given by $0 \to 01$ and $1 \to 0$. Then the substitution matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $(\tau, 1)$ is a left eigenvector of PF-eigenvalue $\tau = \frac{1+\sqrt{5}}{2}$. Giving length τ (or 1) interval to 0 (or 1) letter and letting it A_1 (or A_2) respectively, we can consider the following tile equation

$$\begin{aligned} \tau A_1 &= A_1 \cup (A_2 + \tau) \\ \tau A_2 &= A_1 \end{aligned}$$

and get its suspension tiling. The corresponding MFS is $\Phi = \begin{pmatrix} \{f_1\} & \{f_1\} \\ \{f_2\} & \emptyset \end{pmatrix}$ where $f_1(x) = \tau x$ and $f_2(x) = \tau x + \tau$. We check that the tiling admits overlap coincidence and so the dynamical system has pure point spectrum. See Figure 1.

Example 5.2. (1) Baker-Barge-Kwapisz give a reducible Pisot substitution tiling whose dynamics is not pure discrete in [7, Ex. 5.3]. The substitution is given by $0 \to 051000$, $1 \to 324100$, $2 \to 24100$, $3 \to 324333$, $4 \to 051433$, $5 \to 51433$. The PF-eigenvalue of the substitution matrix is $\alpha = 3 + 2\sqrt{2}$. Let $\beta = \frac{\alpha-1}{4} = \frac{1+\sqrt{2}}{2}$. The tiles are the intervals of lengths $\beta, \beta, 1, \beta, \beta, 1$. The tile equations are given by

$$\begin{aligned} \alpha A_1 &= A_1 \cup (A_6 + \beta) \cup (A_2 + \beta + 1) \cup (A_1 + 2\beta + 1) \cup (A_1 + 3\beta + 1) \cup (A_1 + 4\beta + 1) \\ \alpha A_2 &= A_4 \cup (A_3 + \beta) \cup (A_5 + \beta + 1) \cup (A_2 + 2\beta + 1) \cup (A_1 + 3\beta + 1) \cup (A_1 + 4\beta + 1) \\ \alpha A_3 &= A_3 \cup (A_5 + 1) \cup (A_2 + \beta + 1) \cup (A_1 + 2\beta + 1) \cup (A_1 + 3\beta + 1) \\ \alpha A_4 &= A_4 \cup (A_3 + \beta) \cup (A_5 + \beta + 1) \cup (A_4 + 2\beta + 1) \cup (A_4 + 3\beta + 1) \cup (A_4 + 4\beta + 1) \\ \alpha A_5 &= A_1 \cup (A_6 + \beta) \cup (A_2 + \beta + 1) \cup (A_5 + 2\beta + 1) \cup (A_4 + 3\beta + 1) \cup (A_4 + 4\beta + 1) \\ \alpha A_6 &= A_6 \cup (A_2 + 1) \cup (A_5 + \beta + 1) \cup (A_4 + 2\beta + 1) \cup (A_4 + 3\beta + 1) \end{aligned}$$

(2) B. Sing gives alternative constructions of reducible Pisot substitution tilings which are not pure point in [39, Sec. 6c.3]. We have computed a substitution $0 \to 03$, $1 \to 0$, $2 \to 21$, $3 \to 2$ [personal communication from B. Sing] constructed in the same manner. The PF-eigenvalue of the substitution matrix is τ. The tiles are the intervals of lengths $\tau, 1, \tau, 1$. The tile equations are given by

$$\begin{aligned} \tau A_1 &= A_1 \cup (A_4 + \tau) \\ \tau A_2 &= A_1 \\ \tau A_3 &= A_3 \cup (A_2 + \tau) \\ \tau A_4 &= A_3 \end{aligned}$$

We can confirm that the dynamical systems of the both examples do not admit overlap coincidence.

Example 5.3. Dekking in [9, 10] constructed self-similar tilings from endomorphisms of a free group. Kenyon extended this idea in [17, §6]. We examined Example 7.5 in [41] derived by this method (see Figure 2(a)).

Let $\theta(a) = b$, $\theta(b) = c$, $\theta(c) = a^{-1}b^{-1}$. Later we identify letters a, b, c with vectors starting from the origin defined by $1, \alpha, \alpha^2 \in \mathbb{C}$, where $\alpha \approx 0.341164 + i1.16154$ is the complex root of $\alpha^3 + \alpha + 1 = 0$. We start with words representing three basic parallelograms $[a, b] = aba^{-1}b^{-1}$, $[b, c]$, $[a, c]$. Notice that $\theta([a, b]) = [b, c]$, $\theta([b, c]) = (a^{-1}[a, c]a)(a^{-1}b^{-1}[b, c]ba)$, $\theta([a, c]) = a^{-1}[a, b]a$. Let $U_n = \theta^n([a, b])$, $V_n = \theta^n([b, c])$, and $W_n = \theta^n([a, c])$, where $n \geq 1$. By the above identification, let A_1(or A_2, A_3) be corresponding tiles whose boundaries are given by $\lim_{n\to\infty} \alpha^{-n}U_n$(or $\lim_{n\to\infty} \alpha^{-n}V_n$, $\lim_{n\to\infty} \alpha^{-n}W_n$) respectively. Then these tiles make a self-similar tiling satisfying the tile equation

$$\begin{aligned} \alpha A_1 &= A_2 \\ \alpha A_2 &= (A_2 - 1 - \alpha) \cup (A_3 - 1) \\ \alpha A_3 &= A_1 - 1 \end{aligned}$$

with $\alpha \approx 0.341164 + i1.16154$ which is a root of the polynomial $x^3 + x + 1$. It is known that the corresponding tiling dynamical system is not weakly mixing and has a large discrete part in the spectrum. We check that the dynamical system has pure point spectrum.

We identify $\mathbb{C}$ with $\mathbb{R}^2$ to simplify the notation; the multiplication of α in $\mathbb{C}$ is expressed by the multiplication of a matrix $Q = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ in $\mathbb{R}^2$, where $\alpha = a + bi$. The Delone multi-colour set is given by:

$$\begin{aligned} \Lambda_1 &= \alpha\Lambda_3 - 1 \\ \Lambda_2 &= \alpha\Lambda_1 \cup (\alpha\Lambda_2 - 1 - \alpha) \\ \Lambda_3 &= \alpha\Lambda_2 - 1\,. \end{aligned}$$

We take a basis B of translation vectors $\{1 + \alpha, \alpha + \alpha^2\} \subset \Lambda_2 - \Lambda_2 \subset \Xi(\mathcal{T})$. In Table 1, we write this choice Λ_2 as Colour 2. The MFS is

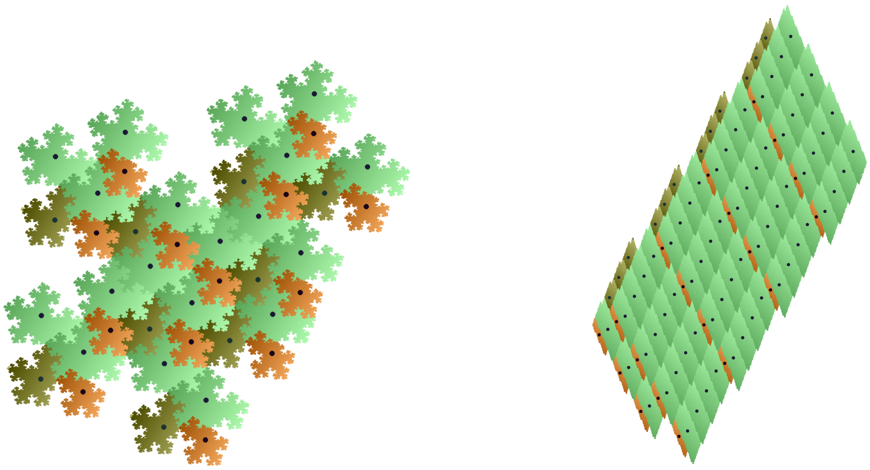
$$\Phi = \begin{pmatrix} \emptyset & \emptyset & \{f_3\} \\ \{f_1\} & \{f_2\} & \emptyset \\ \emptyset & \{f_3\} & \emptyset \end{pmatrix}$$

where $f_1 = \alpha x$, $f_2 = \alpha x - 1 - \alpha$ and $f_3 = \alpha x - 1$. We obtain $\#\mathcal{G}_{coin} = 15$, $\#\mathcal{G}_{res} = 24$[2], $\rho(\mathcal{G}_{coin}) \approx 1.46557$ and $\rho(\mathcal{G}_{res}) \approx 1.32472$. This shows overlap coincidence and therefore the tiling dynamical system associated with this tiling has pure point spectrum. Since this case is self-similar, the Hausdorff dimension w.r.t. the pseudo norm coincides with the usual Hausdorff dimension. So the Hausdorff dimension of the boundary of each tile[3] is $2 \log(\rho(\mathcal{G}_{coin}))/ \log(\rho(\mathcal{G}_{res})) = 1.47131$.

Example 5.4. Continuing Ex. 5.3, we also looked at the self-affine tiling example in [18, Fig. 2 and 3]. Let $\alpha \approx 2.19869$, $\beta \approx -1.91223$ be two real roots of $x^3 - x^2 - 4x + 3 = 0$. The construction is similar to the previous example with a, b, c corresponding to vectors $(1, 1), (\alpha - 1, \beta - 1), (\alpha^2 - \alpha, \beta^2 - \beta)$ in $\mathbb{R}^2$, endomorphisms $\theta(a) = ab$, $\theta(b) = c$, $\theta(c) = ab^4$, and the basic parallelograms $[b, a], [b, c], [a, c]$. Then the self-affine tiling is defined with a

[2] This number depends on other parameters we choose for computation.

[3] The graph $\mathcal{G}_{res}$ is weakly connected and has only one strongly connected component of spectral radius greater than one. Therefore the boundary of each tile has the same dimension. The same holds for all examples in this paper.



(a) Example 5.3 (b) Example 5.4

FIGURE 2

diagonal expansion matrix $Q = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ and the tile equations are

$$\begin{aligned} QA_1 &= (A_2 + \mathbf{1}) \cup A_3 \\ QA_2 &= (A_2 + \mathbf{1}) \cup (A_2 + \mathbf{v}) \cup (A_2 + 2\mathbf{v} - \mathbf{1}) \cup (A_2 + 3\mathbf{v} - 2 \cdot \mathbf{1}) \cup A_3 \\ QA_3 &= (A_1 + \mathbf{v}) \cup (A_1 + 2\mathbf{v} - \mathbf{1}) \cup (A_1 + 3\mathbf{v} - 2 \cdot \mathbf{1}), \end{aligned}$$

where $\mathbf{1} = (1, 1)$ and $\mathbf{v} = (\alpha, \beta)$. It gives overlap coincidence as well. See Table 1 with the notation $\mathbf{v}^2 = (\alpha^2, \beta^2)$. See Figure 2(b).

Example 5.5. Domino tiling is defined with tiles composed of two unit squares. It does not have pure point spectrum [41, Ex. 7.3]. The tiling is shown in Figure 3(a). The expansion map is $Q = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ and the tile equations are given by

$$\begin{aligned} QA_1 &= (A_1 + (-2, 0)) \cup (A_1 + (-2, 3)) \cup (A_2 + (-2, 1)) \cup (A_2 + (-1, 1)) \\ QA_2 &= (A_1 + (-3, 0)) \cup (A_1 + (-3, 1)) \cup (A_2 + (-1, 0)) \cup (A_2 + (-4, 0)). \end{aligned}$$

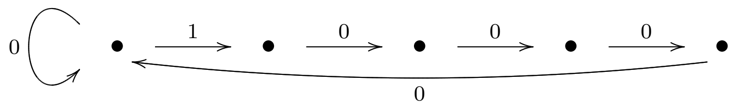
We confirm that the tiling does not admit overlap coincidence.

Example 5.6. In the relation to the explicit construction of Markov partition of toral automorphism, Thurston in [44] introduced $(d-1)$-dimensional self-similar tilings from greedy expansion based on Pisot unit of degree d which is called *Pisot dual tilings*. Their basic properties are studied in Akiyama [1]. Such tiling dynamics are expected to be pure point. We confirm that Pisot dual tilings associated to x^3-x^2-x-1, x^3-x-1, x^3-2x^2-x-1, x^3-3x^2-1 and $x^4-x^3-x^2-x-1$ admit overlap coincidences by our algorithm. We present here two Pisot dual tilings which are 2-dimensional and 3-dimensional.

(1) The minimal Pisot number $x^3 - x - 1$ gives the Hokkaido tiling[4], found in [44] whose expansive factor is $\alpha \approx -0.877439 + i0.744862$ where $1/\alpha$ is a complex root of $x^3 - x - 1$. The corresponding symbolic dynamics is a shift of finite type over two letters $\{0, 1\}$ with

[4] The first author named this after the northern island of Japan.

forbidden words $11, 101, 1001, 10001$, i.e. the letter 1 must be separated by at least four consecutive 0's. Therefore the graph which accepts its language is:



and the associated substitution is

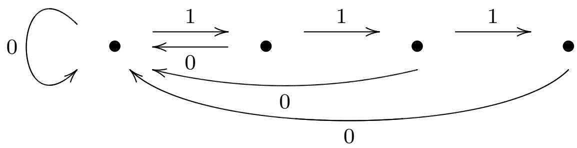
$$a \to ab,\ b \to c,\ c \to d,\ d \to e,\ e \to a.$$

To construct a dual tiling, we reverse arrows of the graph. The MFS Φ is

$$\begin{pmatrix} \{g_0\} & \{g_0\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{g_0\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \{g_0\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \{g_0\} \\ \{g_1\} & \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix}$$

with $g_i = \alpha(x+i)$. See Figure 3(b).

(2) The Pisot dual tiling associated to $x^4 - x^3 - x^2 - x - 1$ is 3-dimensional whose symbolic system is a shift of finite type over $\{0, 1\}$ with a single forbidden word 1111. The associated graph is



whose substitution is

$$a \to ab,\ b \to ac,\ c \to ad,\ d \to a.$$

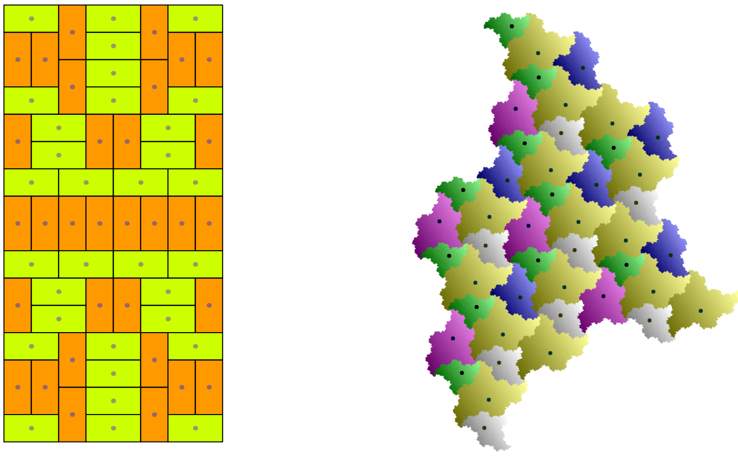
Let us identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. Then the expansion matrix Q and MFS Φ are

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \{g_0\} & \{g_0\} & \{g_0\} & \{g_0\} \\ \{g_1\} & \emptyset & \emptyset & \emptyset \\ \emptyset & \{g_1\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{g_1\} & \emptyset \end{pmatrix},$$

with $\alpha_1 \approx -0.11407 + i1.21675$ and $\alpha_2 = -1.29065$ where $1/\alpha_1, 1/\overline{\alpha_1} \in \mathbb{C}$ and $1/\alpha_2 \in \mathbb{R}$ are three roots of $x^4 - x^3 - x^2 - x - 1 = 0$ of modulus less than one, $g_0 = Qx$ and $g_1 = Q(x+(1,1))$. The translation vectors we choose to compute the overlap coincidence are $(2{\alpha_1}^2 - {\alpha_1}^3, 2{\alpha_2}^2 - {\alpha_2}^3)$, $(2\alpha_1 + {\alpha_1}^2 - {\alpha_1}^3, 2\alpha_2 + {\alpha_2}^2 - {\alpha_2}^3)$, $(2 + \alpha_1 + {\alpha_1}^2 - {\alpha_1}^3, 2 + \alpha_2 + {\alpha_2}^2 - {\alpha_2}^3)$.

Example 5.7. Geometric realization of 1-dimensional substitutions has been studied for a long time, which is motivated by Markov partition of toral automorphism. The original idea came from Rauzy [35] and got extended in a great deal to Pisot substitutions in [2] by Arnoux-Ito. They have a domain exchange structure coming from substitutions and inherit their spectral properties (see also [13]). Recently Arnoux-Furukado-Harriss-Ito in [3] generalized the idea to create a special class of complex Pisot expansion tiling. We examined the example in [3, Prop. 6.8] generated from an automorphism of the free group on four letters:

$$a \to b,\ b \to c,\ c \to d,\ d \to da^{-1}.$$



(a) Example 5.5

(b) Example 5.6. (1)

FIGURE 3

The dual of geometric extention of this automorphism defined in [2] acts on 6 exterior products of 4 fundamental vectors, projected to the contractive plane of:

$$M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

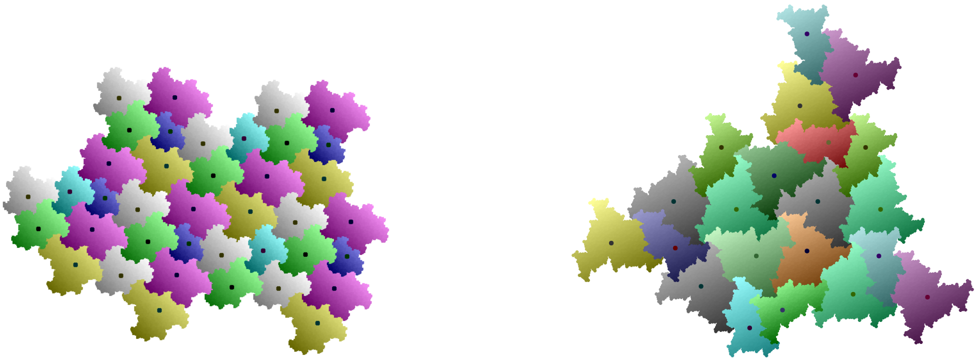
The tile equation for the tiling generated in the contractive plane (identified with $\mathbb{C}$) is

$$\begin{aligned} \alpha A_1 &= A_2 \cup A_3 \\ \alpha A_2 &= A_4 \cup A_5 \\ \alpha A_3 &= A_6 \\ \alpha A_4 &= A_1 \\ \alpha A_5 &= A_2 + \alpha - \alpha^2 \\ \alpha A_6 &= A_4 + 1 - \alpha^2 \end{aligned}$$

with $\alpha \approx -0.727136 + i0.934099$ which is a root of $x^4 - x^3 + 1 = 0$. This tiling no longer has a direct domain exchange structure. Together with the tiling in the expanding plane, it gives an explicit Markov partition of 4-toral automorphism $x \mapsto Mx$. Then MFS Φ is

$$\begin{pmatrix} \emptyset & \emptyset & \emptyset & \{f_1\} & \emptyset & \emptyset \\ \{f_1\} & \emptyset & \emptyset & \emptyset & \{f_3\} & \emptyset \\ \{f_1\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \{f_1\} & \emptyset & \emptyset & \emptyset & \{f_2\} \\ \emptyset & \{f_1\} & \emptyset & \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{f_1\} & \emptyset & \emptyset & \emptyset \end{pmatrix}$$

where $f_1 = \alpha x$, $f_2 = \alpha x + 1 - \alpha^2$ and $f_3 = \alpha x + \alpha - \alpha^2$. The result shows that $\rho(\mathcal{G}_{coin}) \approx 1.40127$ and $\rho(\mathcal{G}_{res}) \approx 1.22074$ and it implies overlap coincidence. We note that the Hausdorff dimension of the boundary of each tile is 1.18242. See Figure 4(a).



(a) Example 5.7 (b) Example 5.8

FIGURE 4

Example 5.8. Bandt-Gummelt in [5] gave a fractal Penrose tiling by fractal kites and darts having exact matching condition. The tile equations are given by

$$\begin{aligned} \tau A_n &= (A_{n+7} + \tau w^n i) \cup (B_{n+4} + \tau w^n c) \cup (A_{n+3} + \tau w^n i) \\ \tau B_n &= (A_{n+7} + \tau w^n i) \cup (B_{n+4} + \tau w^n c) \end{aligned}$$

where $A_n = w^n A_0$ and $B_n = w^n B_0$ for which n is a cyclic index modulo 10, $w = \cos(\pi/5) + i\sin(\pi/5)$, $\tau = \frac{1+\sqrt{5}}{2}$, and $c \in \mathbb{C}$ fulfills $g(\tau^2 i) = i$ with $g(z) = \frac{z}{\tau} w^4 + c$. We confirm that this tiling admits overlap coincidence. Note that the Hausdorff dimension of the boundary of each tile is 1.26634. See Figure 4(b).

Example 5.9. Higher dimension chair tiling is discussed in [24]. We consider a 3-dim chair tiling which is defined by the expansion matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and the MFS Φ is

$$\begin{pmatrix}
\{f_1,f_5\} & \{f_1\} & \{f_1\} & \{f_1\} & \emptyset & \{f_1\} & \{f_1\} & \{f_1\} \\
\{f_2\} & \{f_2,f_6\} & \{f_2\} & \{f_2\} & \{f_2\} & \emptyset & \{f_2\} & \{f_2\} \\
\{f_3\} & \{f_3\} & \{f_3,f_7\} & \{f_3\} & \{f_3\} & \{f_3\} & \emptyset & \{f_3\} \\
\{f_4\} & \{f_4\} & \{f_4\} & \{f_4,f_8\} & \{f_4\} & \{f_4\} & \{f_4\} & \emptyset \\
\emptyset & \{f_5\} & \{f_5\} & \{f_5\} & \{f_1,f_5\} & \{f_5\} & \{f_5\} & \{f_5\} \\
\{f_6\} & \emptyset & \{f_6\} & \{f_6\} & \{f_6\} & \{f_2,f_6\} & \{f_6\} & \{f_6\} \\
\{f_7\} & \{f_7\} & \emptyset & \{f_7\} & \{f_7\} & \{f_7\} & \{f_3,f_7\} & \{f_7\} \\
\{f_8\} & \{f_8\} & \{f_8\} & \emptyset & \{f_8\} & \{f_8\} & \{f_8\} & \{f_4,f_8\}
\end{pmatrix}$$

where $f_1 = Qx + (0,0,0)$, $f_2 = Qx + (1,0,0)$, $f_3 = Qx + (0,1,0)$, $f_4 = Qx + (1,1,0)$, $f_5 = Qx + (1,1,1)$, $f_6 = Qx + (0,1,1)$, $f_7 = Qx + (1,0,1)$, and $f_8 = Qx + (0,0,1)$. This tiling admits overlap coincidence.

Example 5.10. A 3-dimension Thue-Morse tiling can be given by the expanding matrix and the MFS

$$Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \{f_1,f_4,f_6,f_7\} & \{f_2,f_3,f_8,f_5\} \\ \{f_2,f_3,f_5,f_8\} & \{f_1,f_4,f_6,f_7\} \end{pmatrix}$$

where f_i, $1 \le i \le 8$, are given as above in Ex. 5.9. This tiling does not admit overlap coincidence.

Acknowledgments. We thank Boris Solomyak and Benoît Loridant for helpful discussions.

Tiling	Dim.	Colour	Translation vectors	$\#G_{coin}$	$\rho(G_{coin})$	$\rho(G_{res})$	Pure pointedness
Ex. 5.1	1	1	$\tau - 1$	8	1.6180	1	Yes
Ex. 5.2(1)	1	1	$(1+\alpha)/2$	24	5.8284	5.8284	No
Ex. 5.2(2)	1	1	$3 + 4\tau$	20	1.61803	1.61803	No
Ex. 5.3	2	2	$-1-\alpha$ $-\alpha-\alpha^2$	15	1.4656	1.3247	Yes
Ex. 5.4	2	2	$-\mathbf{v}+\mathbf{v}^2$ $-3-3\mathbf{v}+\mathbf{v}^2$	10	4.2044	2.19869	Yes
Ex. 5.5	2	1	$(-3,0)$ $(0,3)$	2	4	4	No
Ex. 5.6(1)	2	1	$2-\alpha^2$ $-2-\alpha+2\alpha^2$	20	1.3247	1.1673	Yes
Ex. 5.7	2	2	$\alpha-\alpha^2$ $1-\alpha^2+\alpha^3$	88	1.4013	1.2207	Yes
Ex. 5.8	2	1	$(\sqrt{v},0)$ $(\sqrt{v}/2,-v/2)$, $v^2-10v+5=0$	751	2.6180	1.8393	Yes
Ex. 5.9	3	1	$(1,1,1)$ $(0,0,2)$ $(0,2,0)$	16	8	4	Yes
Ex. 5.10	3	1	$(1,0,1)$ $(0,1,1)$ $(1,1,0)$	2	8	8	No
Ex. 5.6(2)	3	1	See Ex. 5.6(2)	19	1.9276	1.6234	Yes

Table 1

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